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Is There a Difference Between Aeroacoustics and Aerodynamics? An Aeroelastician's Viewpoint

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Nomenclature

a = speed of sound
c = undisturbed speed of sound, a_∞
c_p = specific heat coefficient at constant pressure
$\boldsymbol{D}$ = strain rate tensor
$\mathrm{D}_c/\mathrm{D}t$ = material time derivative defined in Eq. (33)
D_{ij} = $\frac{1}{2}(v_{i,j} + v_{j,i})$
$\mathrm{D}_W/\mathrm{D}t$ = material time derivative defined in Eq. (28)
$\boldsymbol{d}$ = dissipative term vector; Eq. (41)
$E(\boldsymbol{x})$ = domain function; Eq. (10)
$\mathcal{F}$ = three-dimensional Fourier transform
G = fundamental solution for Laplacian; Eq. (7)
G_A = fundamental solution for acoustics; Eq. (15)
G_{QC} = quasi-potential compressible fundamental solution; Eq. (35)
G_{QS} = quasi-potential subsonic fundamental solution; Eq. (36)
G_{VC} = viscous compressible fundamental solution; Eq. (49)
$H(t)$ = Heaviside step function
h = enthalpy
h_S = slab thickness
J = Jacobian of transformation
k = $\|\boldsymbol{k}\|$
$\boldsymbol{k}$ = Fourier transform frequency vector
L = linear operator
$\mathcal{L}$ = Laplace transform
L^* = linear operator adjoint of L
ℓ = characteristic length
M = undisturbed Mach number, v_∞/c
$\boldsymbol{n}$ = outward normal to $\mathcal{S}$
Pr = Prandtl number, $\mu c_p/\kappa$
p = pressure
$\boldsymbol{q}$ = heat flux vector
R = ideal gas constant
R_S = slab radius
r = $\|\boldsymbol{r}\|$
$\boldsymbol{r}$ = $\boldsymbol{x} - \boldsymbol{y}$
r_β = expression defined by Eq. (37)
S = entropy
$\mathcal{S}$ = surface
s = Laplace transform variable
$\boldsymbol{T}$ = stress tensor
t = time
$\boldsymbol{V}$ = viscous stress tensor
$\mathcal{V}$ = volume
v = $\|\boldsymbol{v}\|$
$\boldsymbol{v}$ = velocity vector
$\boldsymbol{w}$ = vortical velocity vector; Eq. (39)
$\boldsymbol{x}$ = location vector
$\boldsymbol{y}$ = location vector (dummy variable)
α = angle between $\boldsymbol{k}$ and $\boldsymbol{x}$; Eq. (66)
β = $(1 - M^2)^{1/2}$
Γ = circulation
γ = specific heat coefficient ratio, c_p/c_v
$\boldsymbol{\gamma}$ = vortex-layer intensity
Δ = discontinuity across wake ($\Delta f = f_2 - f_1$)
δ = Dirac delta function
δ_ϵ = generic near identity
$\delta_\nu(t)$ = near identity defined by Eq. (50)
$\partial/\partial n$ = $\boldsymbol{n}(\boldsymbol{x}) \cdot$ grad
$\partial/\partial n_y$ = $\boldsymbol{n}(\boldsymbol{y}) \cdot$ grad$_y$
$\boldsymbol{\zeta}$ = vorticity vector
θ = time delay due to wave propagation
θ_C = time delay due to wake convection
ϑ = temperature
$\imath$ = $\sqrt{-1}$
κ = conductivity coefficient ($\boldsymbol{q} = -\kappa$ grad ϑ)
λ = second viscosity coefficient
μ = first viscosity coefficient
ν = μ/ρ_∞

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ν_1 = $(2\mu + \lambda)/\rho_\infty$
ν_2 = $\gamma \nu_1 / 2$
ξ^α = curvilinear coordinates
ρ = density
σ = $-\operatorname{div} \boldsymbol{w}$; Eq. (52)
σ_Q = quasi-potential compressible nonlinear terms
σ_V = viscous compressible higher-order terms
τ = time (dummy variable)
φ = velocity potential, that is, $\boldsymbol{v}$ is equal to $\operatorname{grad}\varphi$
χ = $\boldsymbol{v}_B \cdot \boldsymbol{n}$
$\psi_\nu(r)$ = attenuation factor; Eq. (49)
Ω = solid angle
ω = frequency (oscillatory flows)
ϖ = potential function for $\boldsymbol{d}$, that is, $\boldsymbol{d}$ is equal to $\operatorname{grad}\varpi$

Subscripts

A = acoustics
a = air frame of reference
B = boundary surface
c = constant; Eq. (33)
D = discontinuity
F = fluid
M = material
S = slab
TE = trailing edge
W = wake
0 = initial value
1, 2 = two sides of wake
∞ = undisturbed value

Superscripts

θ = $[\cdots]_{\tau = t - \theta}$
$\sim$ = Laplace transform, that is, $\tilde{f}$ is equal to $\mathcal{L} f$
$\wedge$ = Fourier transform, that is, $\hat{f}$ is equal to $\mathcal{F} f$
$\vee$ = Prandtl–Glauert space; Eq. (38)

I. Introduction

THE paper addresses commonalities and differences between aeroacoustics and aerodynamics. For the sake of simplicity, only exterior flows are considered. In addition, aeroacoustics is understood in its restricted meaning, that is, acoustics generated by aerodynamic flows around moving objects, for example, described by the wave equation for a moving boundary surface.

Even with these limitations, the question is much too broad. Indeed, the answer is in the eyes of the beholder. It really depends on whom you ask. For the man on the street, they do not seem to have anything to do with each other: Acoustics deals with sound and noise and aerodynamics with the air flow around objects, such as cars and airplanes. To emphasize how subjective the answer is, here are some more examples of received replies. For a politician in the Budget Committee for Scientific Research, there is no difference whatsoever. He said, “Aerodynamicists and aeroacousticians are one of a kind. They both want a lot of money for something that to me appears to be totally irrelevant—and I have more important and more vocal constituencies to worry about.” For an informed environmentalist, aerodynamics and aeroacoustics have a lot in common: “They are both useful for the environment: aeroacoustics can contribute to community noise abatement, aerodynamics can help with the reduction of aircraft drag, and hence of chemical pollution.” For the chief executive officer of an aeronautical industry, they have nothing in common: “In aeroacoustics, we want the field noise—in aerodynamics, the aircraft loads. They definitely belong to different branches.” On the other hand, the director of an experimental facility sees the difference in terms of costs: “Aeroacoustics is much more expensive. If you want us to take aeroacoustic measurements, we would have to use the anechoic wind tunnel, and you would have to pay us more—much, much more.”

One would expect more uniform opinions within the scientific community. It is not so! For a continuum-mechanics expert, there is no real difference between aeroacoustics and aerodynamics because they are both governed by the Navier–Stokes equations. On the other hand, a computational fluid dynamicist said, “Once I have solved the aerodynamic problem, the aeroacoustic pressure is just a byproduct; at most, I have to utilize a finer mesh and a larger domain.” You get the opposite view from a traditional aeroacoustician: “Aerodynamics is the easy part. It’s just the input to my code.”

The preceding remarks are offered to emphasize that this paper presents a very personal point of view, that of a theoretical aeroelastician, who in recent years has been trying to become an aeroacoustician, which is, indeed, a much harder task than ever imagined.

What is so different in the point of view of a theoretical aeroelastician, more precisely in the point of view of someone working on flutter? First, note that aerodynamicists traditionally see aerodynamics as a steady-state phenomenon; for flutter, one needs unsteady aerodynamics, which is based on wave propagation and, hence, is closely related to aeroacoustics. Also, in studying unsteady aerodynamics, aeroelasticians typically use boundary integral methods, closely related to those used in traditional exterior aeroacoustics [Ffowcs Williams and Hawkings[1] equation, Kirchhoff method, (see Refs. 2 and 3), etc.]. These factors make the author’s background much closer to theoretical aeroacoustics than to present-day aerodynamics, mostly computational fluid dynamics. Thus, a more specific question will be addressed: Is there something in common between exterior aeroacoustics and unsteady aerodynamics, within the context of boundary integral methods?

The reader is assumed to be familiar with aeroacoustics, but not necessarily with unsteady aerodynamics, in particular, with the issues of potential-flow wakes. Thus, the paper includes a review of the author’s past work in the field of unsteady aerodynamics, with reinterpretation in terms of aeroacoustics. Several new results are included that extend the aerodynamic formulation to aeroacoustics. The paper is introductory in nature, and hence, the mathematical level is kept relatively simple. Accordingly, the emphasis is on wings in uniform translation. For the extension to objects in arbitrary motions, in particular to rotors, see Refs. 4–6.

The paper addresses different types of flows at an increasing level of complexity because certain issues are more easily examined within a simpler context, and it is structured as follows. First, we address some basic concepts of boundary integral methods for potential incompressible flows (Sec. II); in particular, we introduce the main theme of this paper: For every boundary integral equation used to obtain the pressure on the boundary surface (aerodynamic problem), there exists a corresponding boundary integral representation that may be used to obtain the pressure in the field (aeroacoustic problem). Then, we consider acoustics (Sec. III), that is, wave propagation with a boundary surface essentially fixed with respect to the undisturbed air (as distinct from aeroacoustics, which here refers to wave propagation for a moving boundary). This allows us to introduce, at a relatively elementary level, certain mathematical issues related to spurious singular values of the boundary integral equation for acoustics (fictitious eigenvalue difficulty), as well as the corresponding approaches available to remove the problem (along with the time-domain extension). Next, we introduce the formulation for quasi-potential incompressible flows, that is, potential flows with wakes (Sec. IV), and extend it to quasi-potential compressible flows (Sec. V). Then, we turn to viscous compressible flows, for which we present an extension of the aerodynamic formulation to aeroacoustics. This includes the effects on attenuation and dispersion of certain linear viscous terms for the velocity potential, typically neglected in aerodynamics (Sec. VI). Some subtle points of the corresponding numerical implementation are discussed for the limited case of viscous incompressible flows (Sec. VII). A discussion and a summary are given in Secs. VIII and IX, respectively. Some aspects are addressed in the Appendices, not because they are deemed less relevant, but to facilitate reading of the paper. The implications of incompressible-flow assumptions in aeroacoustics are addressed in Appendix A, the value of the function $E(\boldsymbol{x})$ of the boundary surface is discussed in Appendix B; the fundamental solutions for all of the problems addressed are given in Appendix C, whereas the discretization (boundary element method) is outlined in Appendix D.

Note that we normally use $\boldsymbol{x}$ for space and t for time and that $\boldsymbol{y}$ and τ are used as dummy variables. Hence, unless otherwise stated, all of the spatial differential operators, that is, grad, div, curl, and ∇^2, are understood with respect to the variable $\boldsymbol{x}$; similarly, the normal derivative is defined in terms of the variable $\boldsymbol{x}$, as $\partial/\partial n := \boldsymbol{n}(\boldsymbol{x}) \cdot \text{grad}$. To denote the normal derivative with respect to the variable $\boldsymbol{y}$, we use $\partial/\partial n_y := \boldsymbol{n}(\boldsymbol{y}) \cdot \text{grad}_y$, where grad_y is the gradient with respect to the variable $\boldsymbol{y}$. Throughout the paper, $\boldsymbol{n}$ is the outward normal to the surface $\mathcal{S}$.

II. Potential Incompressible Flows

Consider an irrotational flow, that is, a flow with zero vorticity in the whole flowfield. Irrotational flows are potential, that is, there exists a function $\varphi(\boldsymbol{x})$, called the velocity potential, such that

$$\boldsymbol{v}(\boldsymbol{x}) = \text{grad}\,\varphi(\boldsymbol{x}) \qquad (\forall \boldsymbol{x} \in \mathcal{V}_F) \tag{1}$$

where $\boldsymbol{v}(\boldsymbol{x})$ is velocity and $\mathcal{V}_F$ the fluid region.

Combining Eq. (1) with the continuity equation for incompressible flows, $\text{div}\,\boldsymbol{v} = 0$, one obtains

$$\nabla^2 \varphi = 0 \qquad (\forall \boldsymbol{x} \in \mathcal{V}_F) \tag{2}$$

To complete the problem, one needs boundary conditions. We assume the boundary surface $\mathcal{S}_B$, to be impermeable, which implies $\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{v}_B \cdot \boldsymbol{n}$. Combining with Eq. (1) yields the following Neumann boundary condition:

$$\frac{\partial \varphi}{\partial n} = \chi(\boldsymbol{x}) := \boldsymbol{v}_B \cdot \boldsymbol{n} \tag{3}$$

The boundary condition at infinity, in the air frame, that is, a frame of reference rigidly connected to the undisturbed air, is given by $\boldsymbol{v}_\infty = o(r^{-1})$, or using Eq. (1),

$$\varphi = o(1) \tag{4}$$

The problem defined by Eqs. (2–4) is known as the exterior Neumann problem for the Laplace equation and has a unique solution (Kress,[7] pp. 62–67).

Once the problem has been solved, the pressure p is obtained from Bernoulli's theorem (a first integral of the Euler equations), which in the air frame is given by

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} v^2 + \frac{p}{\rho} = \frac{p_\infty}{\rho} \tag{5}$$

Note that the original problem is formulated in terms of four nonlinear differential equations (continuity and Euler equations), for four unknowns (velocity and pressure), with four independent variables (space and time). For potential flows, the formulation is considerably simpler (one linear differential equation for one unknown); the time dependence and the nonlinearities appear only in the evaluation of the pressure via Bernoulli's theorem, where they appear as a derivative and as an algebraic nonlinearity, respectively. Note also that, again for potential flows, one solves for the velocity independently of the pressure: The pressure appears to be "driven" by the velocity. Moreover, potential flows have no memory.

Next, consider a boundary integral formulation for potential flows. Boundary integral methods are based on the use of the so-called fundamental solution G, which, for the Laplace equation, is defined by

$$\nabla^2 G = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{6}$$

subject only to the boundary condition $G = o(1)$ at infinity. This gives (Appendix C.A)

$$G(\boldsymbol{x}, \boldsymbol{y}) = -1/4\pi r \tag{7}$$

with $r = \|\boldsymbol{x} - \boldsymbol{y}\|$.

Multiplying Eq. (2) by $G(\boldsymbol{x}, \boldsymbol{y})$ and Eq. (6) by $\varphi(\boldsymbol{x})$, subtracting, integrating over the fluid volume, using Gauss's divergence theorem (with $\boldsymbol{n}$ being the outward normal) along with Eqs. (3) and (4), and interchanging the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, one obtains the following boundary integral representation for $\varphi(\boldsymbol{x})$:

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left(\chi G - \varphi \frac{\partial G}{\partial n_y} \right) \mathrm{d}\mathcal{S}(\boldsymbol{y}) \tag{8}$$

where $E(\boldsymbol{x}) = 1$ for $\boldsymbol{x} \in \mathcal{V}_F$ and $E(\boldsymbol{x}) = 0$ otherwise. In Eq. (8), $\varphi(\boldsymbol{x})$ on $\mathcal{S}_B$ is not known. However, if $\boldsymbol{x}$ is a smooth point of $\mathcal{S}_B$, one obtains $E(\boldsymbol{x}) = \frac{1}{2}$ (Appendix B). [Throughout the paper we assume, only for the sake of simplicity, that the surface $\mathcal{S}_B$ is smooth. In fact, in the zeroth-order formulation used for the discretization (Appendix D), the collocation points are at the centers of the elements, where the surface is indeed smooth. The case of a nonsmooth point, important for the trailing-edge point in the third-order formulation, is addressed briefly at the end of Appendix B.] Hence,

$$\frac{1}{2}\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left(\chi G - \varphi \frac{\partial G}{\partial n_y} \right) \mathrm{d}\mathcal{S}(\boldsymbol{y}) \qquad (\boldsymbol{x} \in \mathcal{S}_B) \tag{9}$$

Equations (8) and (9) may be combined by setting

$$\begin{aligned} E(\boldsymbol{x}) &= 1 \quad \text{for} \quad \boldsymbol{x} \in \mathcal{V}_F \\ &= \tfrac{1}{2} \quad \text{for} \quad \boldsymbol{x} \in \mathcal{S}_B \\ &= 0 \quad \text{for} \quad \boldsymbol{x} \in \mathbb{R}^3 \backslash \mathcal{V}_F \end{aligned} \tag{10}$$

Contrary to Eq. (8), Eq. (9) is a boundary integral equation, which relates $\varphi(\boldsymbol{x})$ on $\mathcal{S}_B$ to $\chi(\boldsymbol{x}) = \partial\varphi/\partial n$. Note that this equation is a constraint on the Cauchy data of the problem, $\varphi(\boldsymbol{x})$ and $\chi(\boldsymbol{x})$, that must be satisfied by all of the solutions of the Laplace equation.

The solution procedure consists of first solving the boundary integral equation (9). This gives $\varphi(\boldsymbol{x})$ on $\mathcal{S}_B$; then, one may obtain the velocity also on $\mathcal{S}_B$ [the tangential components from $\varphi(\boldsymbol{x})$ on $\mathcal{S}_B$, the normal one from Eq. (3)] and, hence, via Bernoulli's theorem, the pressure still on $\mathcal{S}_B$ (aerodynamic problem). Then, one may use the boundary integral representation (8) to obtain $\varphi(\boldsymbol{x})$ in the field and, hence, via Bernoulli's theorem, the pressure in the field (aeroacoustic problem). (A question arises: Does it make sense to speak of incompressible acoustics? This is addressed in Appendix A, in which the hypotheses behind the incompressible-flow assumption in aerodynamics are also discussed.)

Thus, if one were to ask an expert in boundary integral methods what is the difference between the evaluation of the pressure on the surface (aerodynamic problem) and that in the field (aeroacoustic problem), the reply would be: The same difference that there is between a boundary integral equation and a boundary integral representation.

III. Acoustics

In this section, we extend the boundary integral formulation to classical linear acoustics, that is, we study a potential compressible flow with a boundary that is essentially fixed with respect to the undisturbed air. Contrary to the traditional approach, for example, that of, Colton and Kress[8] and Pierce,[9] we discuss the problem in the time domain, which is more closely related to the physics of the wave-propagation phenomenon. In particular, we will see that this approach to the problem does not require the use of the Sommerfeld condition, which is needed in the frequency-domain approach (see Refs. 8, pp. 65–69, and 9, pp. 177, 178). Then, to introduce the fictitious eigenvalue difficulty (a problem connected to certain spurious singular values of the boundary integral equation for acoustics), we examine the problem in the Laplace transform domain and present a proposal for extending a frequency-domain technique to the time domain to remove the problem.

For consistency, we formulate the problem in terms of the velocity potential. Thus, the velocity is still given by Eq. (1), whereas the pressure is given by the linearized compressible Bernoulli theorem in the air frame [see Eq. (32) and recall that, for isentropic flows, $\mathrm{d}h = \mathrm{d}p/\rho$]

$$\frac{p - p_\infty}{\rho_\infty} = -\frac{\partial \varphi}{\partial t} \tag{11}$$

Classical linear acoustics is governed by the wave equation. [This is a particular case of Eq. (34) and hence, its derivation is not presented here.] Thus,

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = 0 \tag{12}$$

The boundary conditions on $\mathcal{S}_B$ and at infinity are again given by Eqs. (3) and (4), respectively. In addition, we have the initial conditions

$$\varphi(\boldsymbol{x}, 0) = \varphi_0(\boldsymbol{x}), \qquad \dot\varphi(\boldsymbol{x}, 0) = \varphi_1(\boldsymbol{x}) \tag{13}$$

(Note that, for homogeneous initial conditions, the boundary condition at infinity is simply $\varphi = 0$; indeed, we have $\varphi = 0$ for all $\boldsymbol{x}$ having a distance from $\mathcal{S}_B$ greater that ct [see Eq. (17)].) Introduce the fundamental solution for the wave equation, G_A, which is defined by

$$\nabla^2 G_A - \frac{1}{c^2}\frac{\partial^2 G_A}{\partial t^2} = \delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau) \tag{14}$$

with initial conditions $G_A(\boldsymbol{y}, \infty) = \dot G_A(\boldsymbol{y}, \infty) = 0$ and boundary condition $G_A = o(1)$ at infinity.

As shown in Appendix C.B, one obtains

$$G_A = (-1/4\pi r)\delta(t - \tau + r/c) = G\delta(t - \tau + \theta) \tag{15}$$

with $r = \|\boldsymbol{x} - \boldsymbol{y}\|$ and $\theta = r/c$, whereas G is given by Eq. (7). Note that, this expression is obtained without imposing the Sommerfeld radiation condition, which would be required if we were to operate in the frequency domain (see Refs. 8 and 9). Note also that the initial conditions for G_A are at infinity. This is consistent with the general theory of boundary integral equations, whereby the fundamental solution is based on the adjoint operator, Eq. (C9), with boundary-surface conditions excluded; for example, see Refs. 10 and 11. As a consequence, G_A exhibits an anticausal behavior, as necessary because of the change in role between the variables t and τ (to be discussed).

Multiplying Eq. (12) by G_A and Eq. (14) by φ, subtracting, integrating over the fluid volume and over $t \in (0, \infty)$, using Gauss's divergence theorem along with Eqs. (3) and (4), and interchanging the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ as well as t and τ, one obtains the boundary integral representation for the wave equation for $t > 0$:

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}, t) = \int_0^\infty \oint_{\mathcal{S}_B} \left(\chi G_A - \varphi\frac{\partial G_A}{\partial n_y}\right) \mathrm{d}\mathcal{S}(\boldsymbol{y})\,\mathrm{d}\tau - \frac{1}{c^2}\int_{\mathcal{V}_F} \left(\frac{\partial\varphi}{\partial\tau}\, G_A - \varphi\frac{\partial G_A}{\partial\tau}\right) \mathrm{d}\mathcal{V}(\boldsymbol{y})\Bigg|_{\tau=0} \tag{16}$$

Using Eqs. (13) and (15) [which now reads $G_A = G\delta(t - \tau - \theta)$, because of the interchange between t and τ], one obtains

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}, t) = \oint_{\mathcal{S}_B} \left[\chi G - \varphi\,\frac{\partial G}{\partial n_y} + \dot\varphi G\frac{\partial\theta}{\partial n_y}\right]^\theta \mathrm{d}\mathcal{S}(\boldsymbol{y}) + \frac{\partial}{\partial t}(t\phi_0) + t\phi_1 \tag{17}$$

with superscript $\theta =$ subscript $\tau = t - \theta$ and

$$\phi_k(\boldsymbol{x}, t) = \frac{1}{4\pi}\int_\Omega [E(\boldsymbol{y})\varphi_k(\boldsymbol{y})]_{\|x-y\|=ct}\,\mathrm{d}\Omega(\boldsymbol{y}) \tag{18}$$

where Ω is the solid angle for an observer located on $\boldsymbol{x}$. [For the simpler case of $\boldsymbol{x} \in \mathbb{R}^3$, that is, in the absence of the boundary surface, this reduces to Eq. (VII, 2; 37), in Ref. 12, p. 284.]

Again, in Eq. (17), $\varphi(\boldsymbol{x}, t)$ on $\mathcal{S}_B$ is not known. However, if $\boldsymbol{x}$ is a smooth point on $\mathcal{S}_B$, we obtain $E(\boldsymbol{x}) = \frac{1}{2}$ (Appendix B). In this case, Eq. (17) represents a boundary integral equation for the wave equation in the time domain. Again, one solves first the boundary integral equation (17) with $\boldsymbol{x} \in \mathcal{S}_B$ to obtain $\varphi(\boldsymbol{x}, t)$ on $\mathcal{S}_B$ and, hence, the pressure on the boundary (aerodynamic problem). Then, one uses the boundary integral representation [Eq. (17) with $\boldsymbol{x} \in \mathcal{V}_F$] to obtain $\varphi(\boldsymbol{x}, t)$ and, hence, the pressure in the field (acoustic problem).

Observe that the mathematics represent the physical phenomenon in an indirect way. To clarify this statement, consider the sound propagation around a rigid slab, having circular faces with radius equal to R_S and a small thickness h_S. Assume that at time $t = 0$ a point source centered on one of the two circular faces of the slab begins to emit sound (with zero initial conditions). Physics tells us that the sound propagates around the slab and arrives at the center $\boldsymbol{x}_0$ of the opposite side at time $t = (2R_S + h_S)/c$. The earlier mathematical formulation [Eq. (17)] arrives at the same result, but in a very different way. Indeed, the fundamental solution G_A [Eq. (15)] is spherically symmetric and ignores the presence of the boundary. Hence, according to Eq. (17), waves arrive at $\boldsymbol{x}_0$ at time $t = h_S/c$; however, they cancel out exactly, as evidenced, at least for $t < R_S/c$, by the fact that $E(\boldsymbol{x}) = 0$ inside $\mathcal{S}_B$. Indeed, the receiving face may be replaced by a hemispherical surface with radius R_S without affecting the solution. This observation is not presented here just as a matter of curiosity. It has important implication in the discretized formulation: In this case, the cancellation is not exact, and hence, numerical noise (as small as we want, but not zero) arrives at $\boldsymbol{x}_0$ at time $t = h_S/c$.

As already, mentioned there exists a problem (the fictitious eigenvalue difficulty), which is well studied in the frequency domain: It may be shown that in this case there exist certain spurious singular values (related to the natural frequencies of the corresponding interior Dirichlet problem) for which the integral equation is singular (Colton and Kress,[8] pp. 65–107). To the best of the author's knowledge, the problem has not been addressed for the time-domain formulation. To do this, it is convenient to operate with the Laplace transform of the preceding formulation, Eqs. (12–16).

Let a tilde denote the Laplace transform:

$$\tilde\varphi(s) = \mathcal{L}[\varphi(t)] = \int_0^t \varphi(t)\exp(-st)\,\mathrm{d}t \tag{19}$$

The corresponding equations are simply obtained by taking the Laplace transform of Eqs. (12–17). However, note the way in which the initial conditions are dealt with in this case. When it is recalled that $\mathcal{L}\dot u = su - u(0)$ and Eq. (13) is used, the Laplace transform of the wave equation is the nonhomogeneous Helmholtz equation:

$$\nabla^2\tilde\varphi - (s^2/c^2)\tilde\varphi = -(1/c^2)(s\varphi_0 + \varphi_1) \tag{20}$$

The boundary condition on $\mathcal{S}_B$ is $\partial\tilde\varphi/\partial n = \tilde\chi(\boldsymbol{x}) =$ prescribed, whereas at infinity we have $\tilde\varphi = o(1)$. (Note that the Sommerfeld radiation condition need not be imposed in this case either.)

Introduce the fundamental solution $\tilde G^*_A$ defined by [Eq. (C2)]

$$\nabla^2\tilde G^*_A - (s^2/c^2)\tilde G^*_A = \delta(\boldsymbol{x} - \boldsymbol{y}) \tag{21}$$

with $\tilde G^*_A = o(1)$ at infinity. As shown in Appendix C.B, one obtains Eq. (C7), that is,

$$\tilde G^*_A = (-1/4\pi r)\exp(-sr/c) = G\exp(-s\theta) \tag{22}$$

with $r = \|\boldsymbol{x} - \boldsymbol{y}\|$ and $\theta = r/c$.

Multiplying Eq. (20) by $\tilde G^*_A$ and Eq. (21) by $\tilde\varphi$, subtracting, integrating over the fluid volume, using Gauss's divergence theorem, and interchanging the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, one obtains the boundary integral representation for the Helmholtz equation:

$$E(\boldsymbol{x})\tilde\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B}\left(\tilde\chi\tilde G^*_A - \tilde\varphi\,\frac{\partial\tilde G^*_A}{\partial n_y}\right)\mathrm{d}\mathcal{S}(\boldsymbol{y}) - \frac{1}{c^2}\int_{\mathcal{V}_F}\tilde G^*_A(s\varphi_0 + \varphi_1)\,\mathrm{d}\mathcal{V}(\boldsymbol{y}) \tag{23}$$

which coincides with the Laplace transform of Eq. (16). Again, $\tilde\varphi(\boldsymbol{x})$ is not known on $\mathcal{S}_B$. However, if $\boldsymbol{x}$ is a smooth point of $\mathcal{S}_B$, we have $E(\boldsymbol{x}) = \frac{1}{2}$ (Appendix B). Then, assuming, for simplicity, homogeneous initial conditions, the boundary integral equation for the Helmholtz equation is given by

$$\frac{1}{2}\tilde{\varphi}(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left(\tilde{\chi} G - \tilde{\varphi} \frac{\partial G}{\partial n_y} + s\tilde{\varphi} G \frac{\partial \theta}{\partial n_y} \right) \exp(-s\theta)\, \mathrm{d}\mathcal{S}(\boldsymbol{y}) \quad (24)$$

As anticipated (the fictitious eigenvalue difficulty), Eq. (24) is singular for $s = \pm \imath\omega_n$, where ω_n are the natural frequencies of the interior Dirichlet problem for the same boundary surface. The mathematical analysis of the fictitious eigenvalue difficulty requires functional-analysis concepts beyond the mathematical level assumed here; see Colton and Kress,[8] pp. 65–107.

However, a few comments are in order, especially in regard to the time-domain solution. First, note that, if ω approaches one of the ω_n, the solution does not go to infinity because the operator that is applied to $\tilde{\chi}(\boldsymbol{y})$ is also singular; this observation is important for the discretization because in this case the zero on the numerator will occur for a value of ω different for that corresponding to the zero on the denominator, and we do not have exact cancellation of the two zeros. Also, two methods, the composite Helmholtz integral equation formulation (CHIEF)[13] and the composite outward normal derivative overlap relation (CONDOR)[14] are typically used to remedy the problem. CHIEF is a numerical technique [to be used in conjunction with boundary-element collocation methods (Appendix D)], which consists of using additional collocation points located in the interior domain by exploiting the fact that there we have $E(\boldsymbol{x}) = 0$. However, the method fails if the additional control points happen to be located on the nodal surface of the eigenfunction corresponding to the given frequency. This is particularly bothersome if the method is used in conjunction with an optimization code, which performs a repeated sequence of operations without human intervention (see the end of Sec. VIII).

Thus, we concentrate on the CONDOR method. This is a theoretical method, that is, for the nondiscretized operator, and consists of the following. Consider a linear combination of Eq. (24) with the differentiated boundary integral equation, that is, the equation obtained as the limit of the normal derivative of the corresponding boundary integral representation. The coefficients to be used in the linear combination are 1) 1 for Eq. (24) and 2) $\imath\eta$ [with $\eta s_I > 0$, with $s_I = \mathrm{Im}(s)$] for the differentiated boundary integral equation. The resulting boundary integral equation has a unique solution for $\mathrm{Re}(s) \geq 0$ because the spurious singular values are now off the imaginary axis. [For details see Colton and Kress,[8] pp. 103, 104. Note that in Eq. (3.3) of Colton and Kress[8] the condition is $\mathrm{Im}(k) \geq 0$, with $k^2 = \omega(\omega + \imath\gamma/c)$. There, however, the solution is of the type $\exp(-\imath\omega t)$, whereas here we use the Laplace transform, which corresponds to terms of the type $\exp(st)$. Hence, the correspondence $s_I = -\imath\omega$, and $\mathrm{Im}(k) = \mathrm{Re}(s)$.]

The following choice is used in Ref. 15: $\eta = 2$ for $s_I/c < \frac{1}{2}$ (on the basis of Amini[16]), and $\eta = 1/s_I$ for $s_I/c > \frac{1}{2}$ (using the results of Amini and Harris[17] and Kress,[18] which are based on the requirement that the operator be well conditioned as s_I goes to infinity).

However, the use of this function $\eta(s_I)$ makes it problematic to transform back into the time domain. Hence, a different expression for the function $\eta(s_I)$ is proposed here: $\eta(s_I) = s_I/(1 + s_I^2)$, corresponding to a coefficient $s/(1 - s^2)$. This function satisfies the condition $\eta(s_I) >$ for $s_I > 0$. The fact that $\eta(s_I) = 0$ for $s_I = 0$ should not cause problems because this case corresponds to the Laplace equation for which the boundary integral equation (9) is not singular. Also, the behavior at infinity is like $1/s_I$, as suggested in Refs. 16 and 18. Finally, multiplying the equation by $1 - s^2$, we see that the proposed coefficient corresponds, in the time domain, to a linear combination of 1) Eq. (17), 2) its second time derivative, and 3) the first time derivative of the differentiated equation.

From a theoretical point of view, this appears to be a reasonable choice. However, from a practical point of view, this equation might be unnecessarily cumbersome for numerical implementation. To be specific, the discretization yields an approximate operator that has considerably different high-frequency characteristics. Thus, one wonders if it makes any sense to use, for the discretized operator, a function $\eta(s_I)$ that is based on the requirement that the nondiscretized operator be well conditioned. Maybe the simpler $\eta(s_I) = s_I$ (corresponding to the time derivative of the differentiated boundary integral equation) is adequate for the stated objective. In any event, the main issue for the time-domain formulation still remains to be addressed. This pertains the stability of the solution, which requires that the spurious singular values be located to the left of the imaginary axis. In summary, this is considered an open problem that deserves further attention. This appears to be even more so in view of the fact that the numerical discretization presented in Appendix D already moves the spurious singular values to the left of the imaginary axis, at least for values of the time step Δt that are not too small. (Even this lower limit on the value of Δt is apparently removed with the discretization proposed in Ref. 19.) Research activity on the issues addressed in this section is currently underway.

IV. Quasi-Potential Incompressible Flows

Things are not quite as simple in aerodynamics. Indeed, if one recalls the d'Alembert paradox, "steady incompressible potential flows around $\mathcal{S}_B$ produce no force on $\mathcal{S}_B$," it is apparent that one needs a generalization that is capable of predicting lift and drag. This is addressed in this section. See Ref. 20 for details, as well as Ref. 21 for an analysis of the forces produced in the context of quasi-potential flows. As we will see, this generalization includes zero-thickness vortex layers (surfaces of discontinuity for the potential).

For this objective, introduce the material circulation

$$\Gamma_M(t) = \oint_{\mathcal{C}_M(t)} \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{x} = \int_{\mathcal{S}_M(t)} \boldsymbol{n} \cdot \boldsymbol{\zeta}\, \mathrm{d}\mathcal{S}$$

where $\mathcal{C}_M(t) = \partial\mathcal{S}_M(t)$ is a material contour, that is, a contour composed of material points, and recall Kelvin's theorem for incompressible inviscid flows, which states that $\mathrm{d}\Gamma_M/\mathrm{d}t = 0$. Next, consider for simplicity an attached flow around an isolated wing, that is, a flow that leaves the wing surface at a sharp trailing edge, and assume the vorticity $\boldsymbol{\zeta} = \mathrm{curl}\, \boldsymbol{v}$ to be initially vanishing in the whole fluid region $\mathcal{V}_F$, for example, start from rest, so that $\Gamma_M(0) = 0$ for all of the $\mathcal{C}_M$ in $\mathcal{V}_F$. Then, Kelvin's theorem implies $\Gamma_M(t) = 0$ for all of the $\mathcal{C}_M$ that remain in $\mathcal{V}_F$ between 0 and t. This in turn implies that the vorticity vanishes everywhere in the field, except for the locus of the points that came in contact with the trailing edge, for which the preceding assumption (that $\mathcal{C}_M$ remains in $\mathcal{V}_F$ between time 0 and time t) does not apply. These points form a surface, which is called the wake and is denoted by $\mathcal{S}_W$. Hence, the velocity is still given by Eq. (1), which, however, does not hold on $\mathcal{S}_W$:

$$\boldsymbol{v}(\boldsymbol{x}) = \mathrm{grad}\, \varphi(\boldsymbol{x}) \qquad (\forall \boldsymbol{x} \in \mathcal{V}_F \backslash \mathcal{S}_W) \qquad (25)$$

Note that $\Delta\varphi \neq$ const over $\mathcal{S}_W$ implies a tangential discontinuity in the velocity, and, hence, the presence of vorticity, that is, the wake corresponds to a zero-thickness vortex layer. Therefore, these flows are called quasi-potential.

The equation that governs quasi-potential incompressible flows is again the Laplace equation, which, however, does not hold on $\mathcal{S}_W$:

$$\nabla^2 \varphi = 0 \qquad (\forall \boldsymbol{x} \in \mathcal{V}_F \backslash \mathcal{S}_W) \qquad (26)$$

The boundary conditions on the wing and at infinity are the same as for potential flows [Eqs. (3) and (4)]. In addition, we now need boundary conditions on the wake. For a surface of discontinuity, for incompressible flows, the principles of conservation of mass and momentum [Serrin,[22] p. 219, Eq. (54.1)] imply $\boldsymbol{v} \cdot \boldsymbol{n} = \boldsymbol{v}_W \cdot \boldsymbol{n}$ and $\Delta p = 0$, where for any function f, $\Delta f := f_2 - f_1$ is the discontinuity of f across $\mathcal{S}_W$, where the subscripts 1 and 2 are the two sides of the wake at a given point. In turn, these imply

$$\Delta\left(\frac{\partial \varphi}{\partial n}\right) = 0 \qquad (27)$$

$$\frac{\mathrm{D}_W \Delta\varphi}{\mathrm{D}t} := \left(\frac{\partial}{\partial t} + \boldsymbol{v}_W \cdot \mathrm{grad}\right)\Delta\varphi = 0 \qquad (28)$$

where $v_W = \frac{1}{2}(v_1 + v_2)$. Equation (28) is obtained by combining $\Delta p = 0$ with Bernoulli's theorem [Eq. (5)] to yield $\partial\varphi_2/\partial t - \partial\varphi_1/\partial t + \frac{1}{2}(v_1 + v_2)\cdot(v_2 - v_1) = 0$ (see Ref. 23). If one introduces the concept of a wake point x_W as a point having velocity v_W, Eq. (28) implies $\Delta\varphi = \text{const}$ following x_W, or

$$\Delta\varphi(x_W, t) = \Delta\varphi(x_{\mathrm{TE}}, t - \theta_C) \tag{29}$$

where x_{TE} is the trailing-edge point from which x_W originates, whereas $\theta_C = \theta_C(x_W, t)$ is the convection time from x_{TE} to x_W.

Finally, we need a boundary condition at the trailing edge. The Joukowski hypothesis of smooth flow at the trailing edge implies that

$$\lim_{x_W \to x_{\mathrm{TE}}} \Delta\varphi(x_W) = \lim_{x_2 \to x_{\mathrm{TE}}} \varphi(x_2) - \lim_{x_1 \to x_{\mathrm{TE}}} \varphi(x_1) \tag{30}$$

where 2 and 1 here are the sides of the wing surface corresponding to the sides 2 and 1 of the wake, respectively. The discussion of the trailing-edge boundary condition is quite subtle, and the reader is referred to Refs. 24 and 25, where several issues, in particular the relationship between the Kutta condition and the Joukowski hypothesis, are explored in detail. (The main conclusions are reported at the end of Appendix D.)

Next, consider the boundary integral formulation for quasi-potential incompressible flows. Operating as in Sec. II, one obtains that the boundary integral representation for quasi-potential incompressible flows is given by

$$E(x)\varphi(x, t) = \oint_{\mathcal{S}_B} \left(\chi G - \varphi\frac{\partial G}{\partial n_y}\right) \mathrm{d}\mathcal{S}(y) - \int_{\mathcal{S}_W} \Delta\varphi(y_{\mathrm{TE}}, t - \theta_C)\frac{\partial G}{\partial n_y}\, \mathrm{d}\mathcal{S}(y) \tag{31}$$

with $E(x)$ given by Eq. (10). Equation (31) is obtained by using Eq. (8) for the volume outside a surface, $\mathcal{S}_{BW}$, that surrounds the (closed) surface of the body $\mathcal{S}_B$, and the (open) surface of the wake, $\mathcal{S}_W$. Then we take the limit as $\mathcal{S}_{BW}$ tends to $\mathcal{S}_B \cup \mathcal{S}_{W_1} \cup \mathcal{S}_{W_2}$ and use Eqs. (27) and (29) to obtain Eq. (31). In Eq. (31), n on $\mathcal{S}_W$ is pointing from side 1 to side 2.

As before, if $x \in \mathcal{S}_B$, we have $E(x) = \frac{1}{2}$, and Eq. (31) is a boundary integral equation that allows one to solve for $\varphi(x)$ and, hence, for $p(x)$ on $\mathcal{S}_B$ (aerodynamic problem). Then, for $x \in \mathcal{V}_F$, Eq. (31) is a boundary integral representation, which allows one to obtain $\varphi(x)$ and, hence, $p(x)$ in $\mathcal{V}_F$ (aeroacoustic problem).

Note that the wake surface $\mathcal{S}_W$ must be determined as an integral part of the solution by integrating the equation $\mathrm{d}x_W/\mathrm{d}t = v_W(x_W)$, where $v_W(x_W)$ is obtained by taking the gradient of the preceding boundary integral representation (31) (free-wake analysis). In practical applications, the geometry of the wing wake has minor effects on the solution,[23] and hence, it is often prescribed a priori, typically in the direction of the undisturbed flow. However, this is not true for rotors in hover, especially in blade vortex interaction conditions, for which a free-wake analysis is essential.[4,26,27]

V. Quasi-Potential Compressible Flows

In this section, we outline the extension to compressible flows. For details, see Refs. 5, 20, and 28. Consider an inviscid nonconducting fluid and assume the flow to be shock free and initially irrotational and isentropic, for example, at rest and in thermodynamic equilibrium. The governing equations are the continuity equation, the Euler equations, and the entropy transport equation (in the absence of shocks, fully equivalent to the energy equation). Under the preceding assumptions, the flow remains isentropic at all times. Then, $\mathrm{d}p/\rho = \mathrm{d}h$ is an exact differential of a single-valued function, and therefore, Kelvin's theorem applies. Hence, the flow remains irrotational at all times and at all points except for those on the wake [Eq. (25)]. Under these conditions, we have Bernoulli's theorem for compressible, irrotational, isentropic flows, which in the air frame is given by

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}v^2 + h = h_\infty \tag{32}$$

By combining this with the continuity equation in nonconservative form, $\mathrm{D}\rho/\mathrm{D}t + \rho\,\mathrm{div}\, v = 0$, one obtains

$$\nabla^2\varphi = \frac{1}{a^2}\frac{\mathrm{D}_c^2\varphi}{\mathrm{D}t^2} \qquad (x \in \mathcal{V}_{\mathrm{F}}\backslash\mathcal{S}_{\mathrm{W}}) \tag{33}$$

where $a = \sqrt{(\partial p/\partial\rho|_S)}$ is the isentropic speed of sound, whereas $\mathrm{D}_c/\mathrm{D}t := \partial/\partial t + v_c \cdot \mathrm{grad}$. The subscript c indicates that v is kept constant during the second differentiation. In an inertial frame of reference having constant velocity $v_c = v(x, t)$ with respect to the undisturbed flow, Eq. (33) reduces (at point x and time t) to linear wave equation (12).

Note that, for ideal gases with constant specific heat coefficients, we have $h = c_p\vartheta$, and a may be expressed in terms of φ through Bernoulli's theorem [Eq. (32)] by recalling that $a^2 = \gamma p/\rho = \gamma R\vartheta = (\gamma - 1)h$, where $\gamma = c_p/c_v$ is the specific heat coefficient ratio. Then, the velocity potential is the only unknown in the preceding equation. Incidentally, this is the main reason for the author to prefer an aerodynamic–aeroacoustic formulation in terms of velocity potential to one in terms of pressure; also see comments on the Ffowcs Williams and Hawkings[1] equation in Sec. VIII.

In the remainder of this section, we extend the boundary integral formulation for quasi-potential incompressible flows to the case of wings in compressible flows. We take advantage of the fact that even for unsteady flows in practical applications the motion of the wing with respect to a frame of reference in uniform translation (body frame) is typically very small. Thus, we assume the motion of the aircraft surface in the body frame to be infinitesimal. In this case, it is convenient to study the problem in such a frame of reference. Let v_∞ be the velocity of the undisturbed flow with respect to the body frame. For this frame, Eq. (33) may be written as a nonhomogeneous convected wave equation (for both aerodynamics and aeroacoustics):

$$\nabla^2\varphi - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + v_\infty \cdot \mathrm{grad}\right)^2 \varphi = \sigma_Q \qquad (\forall x \in \mathcal{V}_F\backslash\mathcal{S}_W) \tag{34}$$

where $c = a_\infty$, whereas the nonlinear terms σ_Q, important for transonic flows, are treated as nonhomogeneous terms (akin to the Lighthill[29] acoustic analogy). An explicit expression for σ_Q is given in Ref. 30, which gives also a conservative form for σ_Q (obtained from the conservative form of the continuity equation), important in the presence of shocks (shock-capturing scheme).

A question arises: Are the sources real? What is a real source? A loudspeaker consists of a vibrating surface. Are moving surfaces the only real sources? Is a shock wave a real source? Is it the wake? The answer is in the eyes of the beholder. The sources in Eq. (34), or in the quadrupole term in the Ffowcs Williams and Hawkings[1] equation, are typically considered as fictitious sources. However, if we were to make a comparison with experimental results, we would not be able to distinguish between real and fictitious sources, provided that the instrumentation, for instance, a directional microphone, is based on the assumption that the incoming waves are governed by the linear wave equation.

It is easy to see that at infinity the boundary conditions on the body and wake are the same as those for incompressible flows. In addition to the boundary conditions, we need initial conditions: For simplicity, we assume that the fluid is initially at rest, so that $\varphi(x, 0) = \dot{\varphi}(x, 0) = 0$. For nonhomogeneous initial conditions, the formulation is similar to that presented in Sec. III and is given in Ref. 31.

The fundamental solution G_{QC} for quasi-potential compressible flows is obtained by solving the adjoint problem, that is, the differential equation

$$\nabla^2 G_{\mathrm{QC}} - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + v_\infty \cdot \mathrm{grad}\right)^2 G_{\mathrm{QC}} = \delta(x - y)\delta(t - \tau) \tag{35}$$

with initial conditions $G_{\mathrm{QC}}(x, \infty) = \dot{G}_{\mathrm{QC}}(x, \infty) = 0$ and boundary condition $G_{\mathrm{QC}} = o(1)$ at infinity.

For simplicity, we limit ourselves to subsonic undisturbed Mach numbers, $M = v_\infty/c < 1$. For an extension of the formulation to the supersonic case, see Refs. 31 and 32. In this case, the solution of Eq. (35) is given by (Appendix C.C)

$$G_{QS}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = (-1/4\pi r_\beta)\delta(t - \tau + \theta) \tag{36}$$

where, choosing the x_1 axis to be coaligned with $\boldsymbol{v}_\infty$,

$$r_\beta(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{M^2(x_1 - y_1)^2 + \beta^2 r^2} = \sqrt{(x_1 - y_1)^2 + \beta^2\left[(x_2 - y_2)^2 + (x_3 - y_3)^2\right]} \tag{37}$$

with $M = v_\infty/c$, $\beta = \sqrt{(1 - M^2)}$, $r = \|\boldsymbol{r}\| = \|\boldsymbol{x} - \boldsymbol{y}\|$, and $\theta(\boldsymbol{x}, \boldsymbol{y}) = [M(x_1 - y_1) + r_\beta]/\beta^2 c$.

Multiplying Eq. (34) by G and Eq. (35) by φ, subtracting, integrating over $\mathcal{V}_F$ and t, using Gauss's divergence theorem, and interchanging of the variables $\boldsymbol{x}$ and $\boldsymbol{y}$, as well as t and τ, one obtains[5,20,28] the boundary integral representation for Eq. (34):

$$E(\check{\boldsymbol{x}})\varphi(\check{\boldsymbol{x}}, t) = \oint_{\check{\mathcal{S}}_B} \left[\frac{\partial\varphi}{\partial\check{n}_y}\check{G} - \varphi\frac{\partial\check{G}}{\partial\check{n}_y} + \dot{\varphi}\check{G}\frac{\partial\hat{\theta}}{\partial\check{n}_y}\right]^{\check{\theta}} \mathrm{d}\check{\mathcal{S}}(\check{\boldsymbol{y}}) - \int_{\check{\mathcal{S}}_W} \left[\Delta\varphi\frac{\partial\check{G}}{\partial\check{n}_y} - \Delta\dot{\varphi}\check{G}\frac{\partial\hat{\theta}}{\partial\check{n}_y}\right]^{\check{\theta}} \mathrm{d}\check{\mathcal{S}}(\check{\boldsymbol{y}}) + \int_{\check{\mathcal{V}}_F} [\check{G}\sigma_Q]^{\check{\theta}}\, \mathrm{d}\check{\mathcal{V}}(\check{\boldsymbol{y}}) \tag{38}$$

with coordinates $\check{x}_1 = x_1/\beta$, $\check{x}_2 = x_2$, and $\check{x}_3 = x_3$. In addition, $\check{G} = -1/4\pi\check{r}$, with $\check{r} = \|\check{\boldsymbol{x}} - \check{\boldsymbol{y}}\|$, and superscript $\check{\theta}$ subscript $\tau = t - \check{\theta}$, where now (recall the interchange between $\boldsymbol{x}$ and $\boldsymbol{y}$) $\check{\theta} = [M(\check{y}_1 - \check{x}_1) + \check{r}]/\beta c$, whereas $\hat{\theta} = [M(\check{x}_1 - \check{y}_1) + \check{r}]/\beta c$.

Again, for $\boldsymbol{x} \in \mathcal{S}_B$, Eq. (38) is a boundary integral equation that yields φ and, hence, p, on $\mathcal{S}_B$ (aerodynamic problem), whereas for $\boldsymbol{x} \in \mathcal{V}_F$, Eq. (38) is a boundary integral representation that yields φ and, hence, p, in $\mathcal{V}_F$ (aeroacoustic problem).

As mentioned in the Introduction, for the sake of simplicity, the emphasis is on flows around wings in uniform translation. However, the formulation may be extended to surfaces in arbitrary motions, in particular to rotors in maneuvering; see Refs. 5 and 6.

Of course, the fictitious eigenvalue difficulty discussed for acoustics (Sec. III) appears here as well. The time-domain remedy proposed for acoustics applies to this case as well. Again, the time discretization may in itself solve the problem.

VI. Viscous Compressible Flows

In this section, we extend the formulation to viscous compressible flows. For details, see Refs. 33–35. For simplicity, we limit ourselves to ideal gases with constant coefficients. The governing equations are the continuity equation, $\mathrm{D}\rho/\mathrm{D}t + \rho\,\mathrm{div}\,\boldsymbol{v} = 0$, the Navier–Stokes equations, $\rho\mathrm{D}\boldsymbol{v}/\mathrm{D}t = -\mathrm{grad}\,p + \mathrm{Div}\,\boldsymbol{V}$ (where $\boldsymbol{V} = \lambda\,\mathrm{div}\,\boldsymbol{v}\boldsymbol{I} + 2\mu\boldsymbol{D}$), and the entropy transport equation, $\rho\vartheta\mathrm{D}S/\mathrm{D}t = \boldsymbol{V}{:}\boldsymbol{D} - \mathrm{div}\,\boldsymbol{q}$ (where $\boldsymbol{q} = -\kappa\,\mathrm{grad}\,\vartheta$).

The extension to viscous flows of the formulation for quasi-potential flows is obtained through a decomposition of the velocity field into potential and vortical contributions as

$$\boldsymbol{v}(\boldsymbol{x}) = \mathrm{grad}\,\varphi(\boldsymbol{x}) + \boldsymbol{w}(\boldsymbol{x}) \tag{39}$$

where $\boldsymbol{w}(\boldsymbol{x})$ is any particular solution of $\mathrm{curl}\,\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{\zeta}(\boldsymbol{x})$. Indeed, $\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{w}(\boldsymbol{x})$ is irrotational, and hence potential.

The well-known Helmholtz scalar potential-vector potential decomposition, $\boldsymbol{v} = \mathrm{grad}\,\varphi + \mathrm{curl}\,\boldsymbol{\pi}$ (Serrin,[22] p. 165), is a special case of Eq. (39). However, both the scalar potential φ and the vector potential $\boldsymbol{\pi}$ satisfy a Poisson equation, thereby obscuring the wave-like nature of the phenomenon (Refs. 34 and 36). Therefore, the Helmholtz decomposition is of little interest in acoustics.

A different decomposition is used here. To identify suitable requirements for the proposed decomposition, consider first the equation for the potential $\varphi(\boldsymbol{x})$. Combining the Navier–Stokes equations with $\mathrm{d}h = \vartheta\,\mathrm{d}S + \mathrm{d}p/\rho$, and $\mathrm{D}\boldsymbol{v}/\mathrm{D}t = \partial\boldsymbol{v}/\partial t + \frac{1}{2}\,\mathrm{grad}\,v^2 + \boldsymbol{\zeta} \times \boldsymbol{v}$, one obtains

$$\frac{\partial\boldsymbol{v}}{\partial t} + \mathrm{grad}\,\frac{v^2}{2} + \boldsymbol{\zeta} \times \boldsymbol{v} = -\,\mathrm{grad}\,h + \vartheta\,\mathrm{grad}\,S + \frac{1}{\rho}\mathrm{Div}\,\boldsymbol{V} \tag{40}$$

Combining Eqs. (39) and (40) and setting

$$\boldsymbol{d} = \frac{\partial\boldsymbol{w}}{\partial t} + \boldsymbol{\zeta} \times \boldsymbol{v} - \vartheta\,\mathrm{grad}\,S - \frac{1}{\rho}\mathrm{Div}\,\boldsymbol{V} \tag{41}$$

yields $\mathrm{grad}\,(\dot{\varphi} + v^2/2 + h) + \boldsymbol{d} = 0$. This implies $\mathrm{curl}\,\boldsymbol{d} = 0$. Hence, there exists

$$\varpi(\boldsymbol{x}) = \int_\infty^{\boldsymbol{x}} \boldsymbol{d}(\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{y}$$

(with path-independent integral), such that $\boldsymbol{d} = \mathrm{grad}\,\varpi$. In the air frame, combining the preceding equations [$\mathrm{grad}(\dot{\varphi} + v^2/2 + h) + \boldsymbol{d} = 0$ and $\boldsymbol{d} = \mathrm{grad}\,\varpi$] yields a generalized Bernoullian theorem,

$$\dot{\varphi} + v^2/2 + h + \varpi = h_\infty \tag{42}$$

an extension of those considered in Serrin[22] (pp. 153, 168, 260, and 261). Combining this equation with the nonconservative form of the continuity equation, using the equation of state $\rho = \rho(h, S)$, as well as $a^2 := \partial p/\partial\rho|_S = \rho\partial h/\partial\rho|_S$ (isentropic speed of sound), and noting that for ideal gases $D\rho/DS|_h = -\rho/R$ [Ref. 33, p. 4.9, Eq. (4.A.19)], one obtains

$$\nabla^2\varphi + \mathrm{div}\,\boldsymbol{w} = -\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\frac{1}{a^2}\frac{\mathrm{D}h}{\mathrm{D}t} + \frac{1}{R}\frac{\mathrm{D}S}{\mathrm{D}t} = \frac{1}{a^2}\frac{\mathrm{D}}{\mathrm{D}t}\left(\dot{\varphi} + \frac{v^2}{2} + \varpi\right) + \frac{1}{R}\frac{\mathrm{D}S}{\mathrm{D}t} \tag{43}$$

To identify the relevant terms at infinity, that is, the linear terms in φ, because $\boldsymbol{w}$ vanishes exponentially at infinity, note that $\mathrm{Div}\,\boldsymbol{V} = (\lambda + 2\mu)\nabla^2\,\mathrm{grad}\,\varphi$ + higher order terms (HOT). Therefore, $\varpi = -\vartheta_\infty S - \nu_1\nabla^2\varphi + \mathrm{HOT}$, where $\nu_1 = (\lambda + 2\mu)/\rho_\infty$. Hence, $h = h_\infty - \partial\varphi/\partial t + \vartheta_\infty S + \nu_1\nabla^2\varphi + \mathrm{HOT}$. In addition (recalling $h = c_p\vartheta$), $\rho\vartheta\mathrm{D}S/\mathrm{D}t = -\,\mathrm{div}\,\boldsymbol{q} + \mathrm{HOT} = \kappa\nabla^2 h/c_p$. Next, we assume that $Pr = 1/(2 + \lambda/\mu)$. [For zero bulk viscosity this means $Pr = \frac{3}{4}$, which is the diatomic gas value (Serrin,[22] p. 239).] Eliminating S between Bernoulli's theorem and the entropy equation and integrating yields $h = h_\infty - \partial\varphi/\partial t + \mathrm{HOT}$. Combining with the continuity equation and recalling that $a^2 = \gamma R\vartheta$, one obtains

$$\nabla^2\varphi - (1/c^2)\left(\ddot{\varphi} - 2\nu_2\nabla^2\dot{\varphi}\right) = \sigma_V \tag{44}$$

where $c = a_\infty$, $\nu_2 = \gamma\nu_1/2$, whereas σ_V includes all of the HOT.

As already seen for quasi-potential flows, σ_V will appear in a volume integral [Eq. (53)]. On the basis of this consideration, we can identify the requirement for the desired decomposition: Within a boundary integral formulation, it is desirable to reduce as much as possible the region in which $\sigma_V \neq 0$, to minimize the volume of integration, which is computationally expensive. Thus, in the definition of $\boldsymbol{w}(\boldsymbol{x})$, we choose as a requirement that, if at all possible, $\boldsymbol{w}(\boldsymbol{x}) = 0$ in the whole "irrotational region," that is, region in which $\boldsymbol{\zeta}(\boldsymbol{x})$ is computationally negligible, for example, outside boundary layer and wake, for an attached flow past a wing. Strictly speaking, even for initially irrotational flows, $\boldsymbol{\zeta}(\boldsymbol{x})$ and, hence, $\boldsymbol{w}(\boldsymbol{x})$ vanish exponentially at infinity.

To pursue this objective, note that in curvilinear coordinates ξ^α, $\mathrm{curl}\,\boldsymbol{w}(\boldsymbol{x}) = \boldsymbol{\zeta}(\boldsymbol{x})$ is given by

$$J\zeta^1 = \frac{\partial w_3}{\partial\xi^2} - \frac{\partial w_2}{\partial\xi^3}, \qquad J\zeta^2 = \frac{\partial w_1}{\partial\xi^3} - \frac{\partial w_3}{\partial\xi^1}, \qquad J\zeta^3 = \frac{\partial w_2}{\partial\xi^1} - \frac{\partial w_1}{\partial\xi^2} \tag{45}$$

where $J = \partial(x_1, x_2, x_3)/\partial(\xi^1, \xi^2, \xi^3)$ denotes the Jacobian of the transformation $\boldsymbol{x} = \hat{\boldsymbol{x}}(\xi^\alpha, t)$.

Next, choosing $w_1(\xi^1, \xi^2, \xi^3) = 0$ (arbitrarily, but legitimately) and integrating the last two in Eq. (45), we have

$$w_1(\xi^1, \xi^2, \xi^3) = 0$$

$$w_2(\xi^1, \xi^2, \xi^3) = -\int_{\xi^1}^{\infty} J\zeta^3(\breve{\xi}^1, \xi^2, \xi^3)\, \mathrm{d}\breve{\xi}^1$$

$$w_3(\xi^1, \xi^2, \xi^3) = \int_{\xi^1}^{\infty} J\zeta^2(\breve{\xi}^1, \xi^2, \xi^3)\, \mathrm{d}\breve{\xi}^1 \qquad (46)$$

The choice $w_1(\xi^\alpha) = 0$ is legitimate because when it is recalled that $\operatorname{div} \boldsymbol{\zeta} = 0$, it is easy to verify that the preceding solution automatically satisfies the first equation in Eq. (45) as well.

Then, using a suitable direction for the integration, one may obtain $\boldsymbol{w}(\boldsymbol{x}) = 0$ in much (if not all) of the irrotational region (next section).

Next, note an important difference between aeroacoustics and aerodynamics. In aerodynamics, the effects of the linear viscous term are typically very small, and thus, this term is included in σ_V. In this case, we see that the governing equation for φ for viscous flows is formally the same as that for quasi-potential flows (with σ_Q replaced by σ_V). This implies that the same boundary integral formulation used for quasi-potential flows may be used for viscous flows as well. However, in the case of viscous flows, the wake has finite thickness. Thus, there exists no velocity discontinuity. Hence, if $\boldsymbol{w}$ is continuous (a big if; see Sec. VII), then $\Delta\varphi = 0$, and we have

$$E(\check{\boldsymbol{x}})\varphi(\check{\boldsymbol{x}}) = \oint_{\check{\mathcal{S}}_B} \left[\frac{\partial \varphi}{\partial \check{n}_y}\check{G} - \varphi \frac{\partial \check{G}}{\partial \check{n}_y} + \dot{\varphi}\check{G}\, \frac{\partial \hat{\theta}}{\partial \check{n}_y} \right]^{\check{\theta}} \mathrm{d}\check{\mathcal{S}}(\check{\boldsymbol{y}})$$

$$+ \int_{\check{\mathcal{V}}_F} [\check{G}\sigma_V]^{\check{\theta}}\, \mathrm{d}\check{\mathcal{V}}(\check{\boldsymbol{y}}) \qquad (47)$$

On the other hand, in aeroacoustics, the linear viscous term in Eq. (44) is important (at least for distant points) because it yields attenuation and dispersion. This may be seen from the fundamental solution, which, in the air frame, satisfies the equation

$$\nabla^2 G_{\mathrm{VC}} - (1/c^2)\big(\ddot{G}_{\mathrm{VC}} - \nu_2 \nabla^2 \dot{G}_{\mathrm{VC}}\big) = \delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau) \qquad (48)$$

with initial conditions $G_{\mathrm{VC}}(\boldsymbol{x}, \infty) = \dot{G}_{\mathrm{VC}}(\boldsymbol{x}, \infty) = 0$ and boundary condition $G_{\mathrm{VC}} = o(1)$ at infinity. In Appendix C.D, an approximate closed-form solution, still in the air frame, is obtained:

$$G_{\mathrm{VC}}(\boldsymbol{x}, t, \boldsymbol{y}, \tau) = (-1/4\pi r)\psi_\nu(r)\delta_\nu(\tau - t) \qquad (49)$$

where $\psi_\nu(r) = 1 - \exp(-cr/\nu_2)$ is an attenuation factor, whereas

$$\delta_\nu(t) = \frac{cH(t)}{\psi(r)\sqrt{4\pi\nu_2 t}} \left\{ \exp\left[\frac{-(ct - r)^2}{4\nu_2 t}\right] - \exp\left[\frac{-(ct + r)^2}{4\nu_2 t}\right] \right\} \qquad (50)$$

is a near identity, that is, as ν_2 tends to zero, δ_ν tends to the Dirac delta function (Figs. 1 and 2). Indeed, when

$$\int_0^{\infty} \exp\left(-bt - \frac{a}{4t}\right) t^{-\frac{1}{2}}\, \mathrm{d}t = \sqrt{\frac{\pi}{b}} \exp\big(-\sqrt{ab}\big)$$

[Ref. 37, p. 1145, Eq. (17.13.31)] is used, it easy to verify that

$$\int_{-\infty}^{\infty} \delta_\nu(t)\, \mathrm{d}t = 1$$

Hence, as ν_2 tends to zero, G_{VC} [Eq. (49)] tends to the fundamental solution of the inviscid wave equation G_A [Eq. (15)].

Still, in aeroacoustics, for $\nu_2 \neq 0$, the boundary integral representation is easily obtained using the procedure outlined earlier: Note that the terms in Eq. (44) that include a time derivative do not contribute to the boundary integral representation for φ [Eq. (47)]

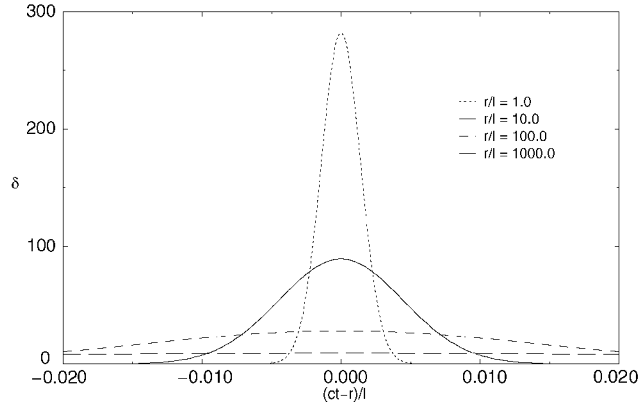

Fig. 1 For $c\ell/\nu_2 = 10^6$ and $r/\ell = 1, \ldots, 1000$, $\delta = c\delta_\nu/\ell$ as a function of $(ct - r)/\ell$.

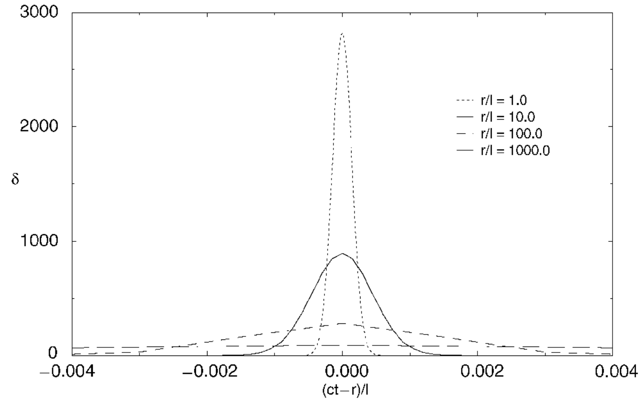

Fig. 2 For $c\ell/\nu_2 = 10^8$ and $r/\ell = 1, \ldots, 1000$, $\delta = c\delta_\nu/\ell$ as a function of $(ct - r)/\ell$.

because of the homogeneous initial conditions on φ and G_{VC}; the boundary terms multiplied by ν_2 are typically very small and may be neglected. Then, the primary difference is due to the fundamental solution. For simplicity, let us limit the discussion to the fundamental solution given by Eq. (49). It is apparent from Figs. 1 and 2 that, for high Reynolds numbers and for nondistant points, the fundamental solution is very close to that of the quasi-potential flows [also Eq. (49)]. Thus, for ν_2 very small, the formulation for aeroacoustics is also given by Eq. (47), provided that the superscript $\check{\theta}$ is interpreted as an average retarded value (convolution with the near identity δ_ν) and that G is multiplied by the attenuation factor $\psi(r)$. Of course, for very high Reynolds numbers and for really nondistant points, one recovers the aerodynamic formulation (47). Note that in the preceding discussion, we implicitly assumed to be working in the body frame; to accomplish this, we need to make the same transformation used in Appendix C.C, which is dictated, but not limited, by the presence of the Dirac delta function.

Moreover, there is an important issue that should be discussed. We have assumed that $\Delta\boldsymbol{w}(\boldsymbol{x}) = 0$. Under what condition is this true? For the sake of clarity, in discussing the requirements for having $\boldsymbol{w}(\boldsymbol{x})$ continuous, we restrict the problem to incompressible flows (Sec. VII) because the generalization to compressible flows is apparent.

In Sec. VII, we also address the issue of the use of Eq. (47) for $\boldsymbol{x} \in \mathcal{S}_B$. We will see that, in this case, the main unknown in this equation is the vorticity at the boundary because for viscous flows the velocity at the boundary is prescribed.

Finally, consider a novel derivation of the expression for $\boldsymbol{w}(\boldsymbol{x})$, which clarifies the meaning of $\varphi(\boldsymbol{x})$ in Eq. (39). Using the well-known identity $e_{\alpha\beta\gamma}\, e^{\gamma\sigma\tau} = \delta^\sigma_\alpha \delta^\tau_\beta - \delta^\tau_\alpha \delta^\sigma_\beta$, where $e_{\alpha\beta\gamma}$ and $e^{\gamma\sigma\tau}$ denote the permutation symbol, and [see Eq. (46)] $J\zeta^\gamma = e^{\gamma\sigma\tau} v_{\sigma,\tau}$, where $f_{,\tau} = \partial f/\partial \xi^\tau$, we have $e_{\alpha\beta\gamma} J\zeta^\gamma = (\delta^\sigma_\alpha \delta^\tau_\beta - \delta^\tau_\alpha \delta^\sigma_\beta) v_{\sigma,\tau} = v_{\alpha,\beta} - v_{\beta,\alpha}$.

Multiplying by $\delta_1^\beta\,\mathrm{d}\xi^1$ and integrating over a ξ^1 line from infinity to $\boldsymbol{x}$, denoted by $\mathcal{L}(\infty, \boldsymbol{x})$, one obtains [see Eq. (46)] $w_\alpha = v_\alpha - \varphi_{,\alpha}$, in agreement with Eq. (39), where now

$$\varphi(\boldsymbol{x}) = \int_{\mathcal{L}(\infty,\boldsymbol{x})} \boldsymbol{v}(\boldsymbol{y}) \cdot \mathrm{d}\boldsymbol{y} \tag{51}$$

Thus, $\varphi(\boldsymbol{x})$ is the line integral of the velocity over the same path used for $\boldsymbol{w}(\boldsymbol{x})$. Note that, contrary to the potential-flow case, the integral in Eq. (51) is path dependent. Note also that the definition of $\varphi(\boldsymbol{x})$ in Eq. (51) depends only on the integration paths, $\mathcal{L}(\infty, \boldsymbol{x})$. Hence, $\boldsymbol{w}(\boldsymbol{x})$ is not as strongly dependent on the choice of the curvilinear coordinates ξ^α as it seems from Eq. (46), although its covariant components are, of course, ξ^α dependent.

VII. Viscous Incompressible Flows

In this section, we discuss the issue of the continuity of $\boldsymbol{w}(\boldsymbol{x})$. In the process, we obtain a better appreciation for the subtleties of the numerical implementation, as well as a clarification of the earlier statement, that is, that using a suitable direction for the integration one obtains $\boldsymbol{w} = 0$ in much, if not all, of the irrotational region. (For details, see Refs. 34, 35, and 38.) Let us assume that, at a given time step, the vorticity is known, for example, obtained by step-by-step integration of the vorticity transport equation, discretized, for instance, by finite differences. Thus, we can concentrate on the equation for the potential φ.

Assume, for a moment, $\boldsymbol{w}(\boldsymbol{x})$ to be continuous across the wake surface; then $\Delta\varphi = \Delta(\partial\varphi/\partial n) = 0$ because for viscous flows $\boldsymbol{v}(\boldsymbol{x})$ is continuous. Combining Eq. (39) with the continuity equation for incompressible flows, $\operatorname{div} \boldsymbol{v} = 0$, and setting $\sigma = -\operatorname{div} \boldsymbol{w}$, one obtains

$$\nabla^2\varphi = \sigma \qquad (\boldsymbol{x} \in \mathcal{V}_F) \tag{52}$$

The boundary integral representation is

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left(\frac{\partial\varphi}{\partial n_y} G - \varphi \frac{\partial G}{\partial n_y} \right) \mathrm{d}\mathcal{S}(\boldsymbol{y}) + \int_{\mathcal{V}_F} G\sigma\, \mathrm{d}\mathcal{V}(\boldsymbol{y}) \tag{53}$$

The boundary condition on the boundary surface $\mathcal{S}_B$, assumed to be impermeable, is $\boldsymbol{v}\cdot\boldsymbol{n} = \partial\varphi/\partial n + \boldsymbol{w}\cdot\boldsymbol{n} = \boldsymbol{v}_B\cdot\boldsymbol{n} =: \chi$ [Eq. (3)], or

$$\frac{\partial\varphi}{\partial n} = \chi - \boldsymbol{w}\cdot\boldsymbol{n} \tag{54}$$

The issue of the continuity of $\boldsymbol{w}$ depends on the direction of integration used for $\boldsymbol{w}$. Two options are considered in the following. In both cases, the presence of the wake makes it convenient to use a C grid around a cross-section of the wing. Here, a given ξ^1 line on $\mathcal{S}_B$ is assumed to close on itself at the trailing edge and then to cover the two sides of $\mathcal{S}_D$ over the same line. Also, for simplicity, we assume the flow to be almost potential. (Following Chorin and Marsden,[39] p. 60, this is a flow in which the vorticity is confined to some thin layers of fluid, such as boundary layer and wake.)

In the first option (considered in Ref. 38 and hereby referred to as option A), one chooses the integration coordinate in Eq. (46), ξ^1, to coincide with the normal to the body and to a surface $\mathcal{S}_D$ emanating from the trailing edge (midsurface of the wake, which is analogous to $\mathcal{S}_W$ for quasi-potential flows). This particular integration coordinate is hereby denoted by η. (The surface $\eta = 0$ covers $\mathcal{S}_B$ and the two sides of $\mathcal{S}_D$.) This implies $\boldsymbol{w}\cdot\boldsymbol{n} = w_1 = 0$. Thus, if one still assumes that $\boldsymbol{w}$ is continuous so that $\Delta\varphi = \Delta(\partial\varphi/\partial n) = 0$ across $\mathcal{S}_D$, Eq. (53) integrated by parts in the direction $\xi^1 = \eta$ yields

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left[(\chi + \chi_1)G - \varphi \frac{\partial G}{\partial n_y} \right] \mathrm{d}\mathcal{S}(\boldsymbol{y}) + \int_{\mathcal{V}_F} \frac{\partial G}{\partial \eta}\sigma_1\, \mathrm{d}\mathcal{V}(\boldsymbol{y}) \tag{55}$$

where

$$\sigma_1 = \frac{1}{\sqrt{a}} \int_\eta^\infty \sqrt{a}\sigma\, \mathrm{d}\check{\eta} \tag{56}$$

$$\chi_1 = \frac{1}{\sqrt{a}} \int_0^\infty \sqrt{a}\,\sigma\, \mathrm{d}\check{\eta} \tag{57}$$

Note that, for almost potential flows, $\boldsymbol{w}(\boldsymbol{x})$ and, hence, $\sigma(\boldsymbol{x})$ vanish in the irrotational region. Also, within the boundary-layer approximations, the preceding expression for χ_1 coincides with the Lighthill[40] transpiration velocity. (See Ref. 38 for details.) Thus, this may be considered as an exact generalization of the classical potential-flow/boundary-layer coupling. However, a problem arises: Because of the C grid, $\boldsymbol{w}$ is discontinuous across $\mathcal{S}_D$, unless $J\boldsymbol{\zeta}$ is antisymmetric with respect to ξ^1, for example, symmetric flows. Thus, except for this last case, the formulation is cumbersome to implement. We would need a doublet (dipole) distribution to compensate for the tangential discontinuity of $\boldsymbol{w}$ on $\mathcal{S}_D$; in addition, we would have a source (monopole) distribution due to the transpiration velocity [Eq. (57)] on the $\mathcal{S}_D$, arising from the integration by parts. Both terms vanish for symmetric flows with symmetric curvilinear coordinate system.

In the second option (considered in Ref. 34, and hereby referred to as option B), one chooses the integration coordinate ξ^1 to be essentially in the direction of the flow. (The surfaces $\mathcal{S}_B$ and $\mathcal{S}_D$ are covered by ξ^1 lines). Then, if one still assumes for a moment that $\boldsymbol{w}(\boldsymbol{x})$ is continuous so that $\Delta\varphi = \Delta(\partial\varphi/\partial n) = 0$ across the wake, and uses Eq. (54) and Gauss's theorem, Eq. (53) yields

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left(\chi G - \varphi \frac{\partial G}{\partial n_y} \right) \mathrm{d}\mathcal{S}(\boldsymbol{y}) + \int_{\mathcal{V}_F} \boldsymbol{w}\cdot \operatorname{grad} G\, \mathrm{d}\mathcal{V}(\boldsymbol{y}) \tag{58}$$

To interpret this equation, consider an almost potential flow, as defined earlier. For the sake of clarity, assume that the coordinate system ξ^α is such that each vortex line is within a surface $\xi^3 = \text{const}$. Then, $\zeta^3 = 0$, and Eq. (46) yields $w_1 = w_2 = 0$. Thus, $\boldsymbol{w} = w_3\,\boldsymbol{g}^3$, or $\boldsymbol{w} = w\,\boldsymbol{n}$, with $\boldsymbol{n} = \boldsymbol{g}^3/\|\boldsymbol{g}^3\|$ being the unit normal to the surfaces $\xi^3 = \text{const}$. In this case, $\boldsymbol{w}\cdot\operatorname{grad} G = w\partial G/\partial n$, and the volume integral may be seen as a distribution of diffused doublets (dipoles), in direction of $\boldsymbol{n}$. In the limit, as the vortex–layer thickness goes to zero, the almost potential flow tends to a quasi-potential flow, and it may be shown that from Eq. (58) we recover the boundary integral representation for incompressible quasi-potential flows, Eq. (31) (Ref. 34, pp. 26–30). Note that, also in this case, for almost potential flows, $\boldsymbol{w}(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ vanish in the entire irrotational region. Again, a problem arises: Here $\boldsymbol{w}$ is discontinuous, unless $J\boldsymbol{\zeta}$ is symmetric with respect to η, for example, antisymmetric flow with symmetric curvilinear coordinate system. Thus, except for this last case, the formulation is cumbersome to implement.

A remedy to the preceding two problems is proposed in Ref. 35: Decompose the field $J\boldsymbol{\zeta}$ around an isolated wing into two contributions, one symmetric and the other antisymmetric with respect to the direction η, and then use option A for the antisymmetric contribution and option B for the symmetric one. This yields

$$E(\boldsymbol{x})\varphi(\boldsymbol{x}) = \oint_{\mathcal{S}_B} \left[\left(\chi + \chi_1^A\right)G - \varphi \frac{\partial G}{\partial n_y} \right] \mathrm{d}\mathcal{S}(\boldsymbol{y}) + \int_{\mathcal{V}_F} \left(\sigma_1^A \frac{\partial G}{\partial \eta} + \boldsymbol{w}^B \cdot \operatorname{grad} G \right) \mathrm{d}\mathcal{V}(\boldsymbol{y}) \tag{59}$$

where the superscripts A and B refer to option A and B, respectively. However, this is too limiting for more complex configurations, for instance, for a low-wing/high-tail combination or for high-lift devices.

A refinement is proposed here. Note that, within option B, the symmetry requirement on $\boldsymbol{\zeta}$ need be applied only on the surface. Thus, for each cross section in a compact domain contained within $\mathcal{V}_F$, consider an antisymmetric continuation of the antisymmetric portion of the vorticity on the boundary. Treat these continuations with option A (which works for antisymmetric $J\zeta$). Subtract these from the vorticity field; the remaining vorticity field is symmetric over each boundary and may be treated with option B.

Finally, a few words regarding the use of Eq. (53) for $x \in \mathcal{S}_B$. In viscous flows, the boundary condition is $v = v_B$, that is, not only no penetration, but also no slip. Thus, in addition to Eq. (54), we have $n \times \text{grad}\,\varphi = n \times (v_B - w)$, which allows one to obtain φ on $\mathcal{S}_B$ (except for an arbitrary additive constant) from v_B and w. Hence, the real unknown in this case is the vorticity at the boundary nodes. (As mentioned earlier, the vorticity at the field nodes may be evaluated by finite differences, provided that the values at the boundary nodes are known.) To this end, one may use Eq. (59) to determine φ on $\mathcal{S}_B$. In general, this will produce slip at the boundary, which is to be interpreted as a vortex layer with intensity $\gamma = -n \times (\text{grad}\,\varphi + w - v_B)$. As this diffuses, the slip is removed, and hence, the no-slip boundary condition is automatically satisfied. Thus, quasi-potential flows are very close to viscous flows, not only numerically, but also conceptually, provided that the discontinuity between fluid and solid is seen as a zero-thickness vortex layer within the flow. (See Refs. 34 and 36 for a deeper analysis of this point.) This issue is currently under investigation.

VIII. Discussion

From the preceding considerations, we can say that, within the context of boundary integral methods, the differences between exterior aeroacoustics and unsteady aerodynamics are minor (especially for nondistant points). Indeed, the two disciplines are closely interrelated and further analysis of the relationship is warranted. In particular, we have examined the relationship only for the formulation in nonprimitive variables, that is, within the context of boundary integral equations for the velocity potential and the extension to viscous flows through the decomposition of Eq. (39). Thus, the presented analysis is by no means complete. Nothing has been said about boundary integral formulations for primitive variables. These have been examined by Piva and Morino[10,11] for unsteady viscous compressible flows, for the limited case of $Pr = 0$; the more general case has been considered in Ref. 41. These papers (Refs. 10, 11, and 41) are formulated for acoustics, that is, for an air frame; the transformation of Appendix C.C may be applied to shift to aeroacoustics.

Note that the Ffowcs Williams and Hawkings[1] formulation is recovered as a particular case of that of Refs. 10, 11, and 41. Further analysis of these issues is now under consideration and will be the subject of a separate paper. However, a few remarks on the relationship with the two approaches typically used in aeroacoustics, that of Ffowcs Williams and Hawkings[1] and the Kirchhoff method (see Refs. 2 and 3) are appropriate. As mentioned, the present approach is preferred by the author because, for quasi-potential flows, there exists an equation [Eq. (33)] that is self-contained in the sense that the only unknown is φ. This provides us with a unified approach for aerodynamics and aeroacoustics, which allows one to explore the relationship between the two disciplines. This is not true for the other two approaches used in aeroacoustics, that of Ffowcs Williams and Hawkings[1] and the Kirchhoff method (see Refs. 2 and 3), where the formulation contains quantities (specifically, in the Lighthill tensor) that are assumed to be obtained independently. Note that if one considers the exact (nonlinear) formulation, these two methods are fully equivalent because they may be obtained from each other through an integration by parts. (For example, see Refs. 42–44, which include permeable surfaces for both methods.)

Note also that there exist problems even if we limit ourselves to the linear quasi-potential formulation. These are due to the presence of the wake. To see this, note that, in the Ffowcs Williams and Hawkings[1] method for quasi-potential flows, assuming for simplicity that the wake is flat and parallel to the flow, the wake integrals are expressed in terms of the discontinuity of p and $v \cdot n$, quantities that are continuous across the wake [Eq. (27)]. Thus, within the linear formulation, there is no contribution from the wake, across which these quantities are continuous. Indeed, the contribution from the wake arises from the quadrupole,[45] and hence, it is a nonlinear effect. This is a major advantage in the use of the boundary integral representation (acoustic problem) because the contribution from the wake is computationally expensive. However, it causes problems in the use of the boundary integral equation (aerodynamic problem) because the preceding consideration implies that the solution is not unique (because both the potential and the quasi-potential solution satisfy this equation). Thus, on the basis of the Fredholm-alternative theorem (see Colton and Kress,[8] p. 22, and note that the operator is compact) the integral operator is singular and cannot be used to find φ on $\mathcal{S}_B$ (aerodynamic problem). On the other hand, in the Kirchhoff method (see Refs. 2 and 3), there is a contribution from the wake (which to the best of the author's knowledge has never been included in the computations) even if the Lighthill tensor is neglected because in general, $\partial p/\partial n$ is discontinuous, as may be seen from the Euler equations. However, if $\partial n/\partial t = 0$, for example, for steady problems, the discontinuity is nonlinear; in addition, if n is constant, for example, flat wake, then $\Delta(\partial p/\partial n) = 0$. These issues are more complicated if the wake is not parallel to the flows because the normal to S_W differs from the normal in the Prandtl–Glauert space; this is even more so in the case of rotors.

The preceding remarks should not be construed as to imply that the Ffowcs Williams and Hawkings[1] and the Kirchhoff (see Refs. 2 and 3) methods are computationally less convenient than the present method, only that they do not provide a unified approach to the two problems. Indeed, the presence of $\Delta\varphi$ in the formulation presented here could be a major problem in aeroacoustics. It is not a problem if the wake is straight and we are operating in the frequency domain because in this case the wake contribution may be integrated analytically along the flow. On the other hand, in aerodynamics the presence of the wake may be an advantage because it allows one to take into account the effects of wake rollup (important, for instance, for helicopter rotors in hover).

Hence, another question arises: Is the formulation presented useful as a computational methodology? A few words on the numerical implementation and validation appear in order. First, note that most of the panel-method codes developed to design aircraft are based on the present formulation for quasi-potential flows (e.g., Ref. 46). Note that two types of discretizations have been used, hereby referred to as the zeroth-order and the third-order formulation. (For details, see the end of Appendix D.) The zeroth-order quasi-potential numerical formulation has been implemented and validated for unsteady three-dimensional compressible flows. (Unless otherwise stated, see Refs. 4, 43, and 44, which present overviews of the zeroth-order numerical results available.) Specifically, 1) for subsonic flows (frequency and time domain) the validation includes complex configuration, for example, wing–body–tail, and innovative configurations (even multiply connected, e.g., box wing[47]), as well as rotors,[4–6] and 2) for transonic flows (time domain) and supersonic flows (frequency domain[31,32]) the validation is limited to isolated wings.

The third-order formulation is more recent: It has a higher rate of convergence, but it requires a more complex geometry description, for example, base vectors; however, such a formulation is essential if the differentiated boundary integral equation, which has a hypersingular kernel, is to be used. (See considerations of the CONDOR method in Sec. III, as well as Ref. 15, which uses this approach for the acoustic scattering around a sphere.) The validation of the third-order formulation for quasi-potential flows is limited to unsteady incompressible three-dimensional flows around isolated wings. (See Ref. 25, which includes innovative configurations, that is, box wings.) The extension to quasi-potential compressible flows [Eq. (38)] is nearly completed.

The viscous-flow formulation is very recent. The validation of option A (Sec. VII) is presented in Ref. 48 and is limited to two-dimensional incompressible flows. For the combination of options A and B, recent, as yet unpublished, results confirm the validity of the formulation. As mentioned already, the issue of the boundary condition for the vorticity is under investigation. Option A has also been used in Ref. 49 to study transonic Euler flows, with vorticity generated by the shock. In addition, the extension in Ref. 38 of Lighthill's[40] transpiration velocity has been used to study the potential-flow/boundary-layer interaction for three-dimensional subsonic flows around an isolated wing, in particular, a box wing,[47,50] and for two-dimensional transonic flows around an isolated airfoil.[43,44]

Moreover, in Ref. 51, the two-dimensional boundary-layer equations have been reinterpreted in term of vorticity. Specifically, integrating twice along the boundary-layer thickness in the vorticity–transport equation, one obtains an equation that is formally equal to the boundary-layer equation, with the displacement and momentum thicknesses defined in terms of vorticity. These definitions reduce to the classical ones in terms of velocity if the boundary-layer assumptions are applicable. The three-dimensional extension is currently underway.

Finally, the described formulations are being incorporated in a multidisciplinary-optimization code for preliminary design of innovative configurations, for which only first-principle-based methods are available.[50,52] The methodology presented here is particularly useful in this context because it provides accurate (and refinable) results with relatively low computational costs.

IX. Summary

Let us go back to the original question: Is there a difference between exterior aeroacoustics and unsteady aerodynamics? Within the context of boundary integral methods that are based on the velocity potential (with its generalization to viscous flows), the main theme of this paper is that for every boundary integral equation used in aerodynamics to obtain φ (and, hence, velocity and pressure) on the boundary surface, there exists a corresponding boundary integral representation that afterward may be used to evaluate φ (and, hence, velocity and pressure) in the field, that is, for aeroacoustics.

Also, some new results were presented that show some differences between the two disciplines. In particular, in Sec. VI, we presented an extension to aeroacoustics of the aerodynamic formulation for viscous compressible flows of Refs. 20, 28, and 33–35. Specifically, for distant points, linear viscous terms exist in the equation for φ, which are typically neglected in aerodynamics and which in aeroacoustics must be included, at least for distant points, because they yield attenuation and dispersion. In addition, we obtained an explicit closed-form expression for the viscous compressible fundamental solution for the case $Pr = \gamma/(2+\lambda/\mu)$ (Appendix C.D). Furthermore, the formulation of Ref. 35 for removing discontinuities on $\boldsymbol{w}$ has been extended to more general configurations. Moreover, a novel derivation of Eq. (46) has been presented at the end of Sec. VI that clarifies the meaning of $\varphi(\boldsymbol{x})$ in Eq. (39) and clarifies the extent of the dependence of $\boldsymbol{w}(\boldsymbol{x})$ on the the choice of the coordinate system.

Additional novel results include 1) a proposal for the time-domain remedy to the fictitious eigenvalue difficulty (Sec. III) and 2) the formulation of Appendix C.B, which yields the correct causal behavior of the solution without the need for invoking the Sommerfeld radiation condition. Finally, in Appendix A, we show that, if $M \ll 1$ and $\omega\ell/c \ll 1$, the incompressible-flow assumption may be used in aeroacoustics as well, provided that the signal is delayed by $\Delta t = \|\boldsymbol{x} - \boldsymbol{y}_0\|/c$. (Of course this difference is relevant only for distant points.)

Advantages and limitations of the formulation have been discussed in Sec. VIII, along with the numerical validation and work in progress.

Finally, note that the decomposition in Eq. (39) has theoretical advantages that have not been discussed: For instance, the decomposition into potential and vortical velocity allows one to separate noise, that is, randomness in the potential velocity, and turbulence, that is, randomness in the vortical velocity. That the turbulence is connected with the vorticity is apparent in incompressible flows because for harmonic functions the value at a point $\boldsymbol{x}$ coincides with the average over a spherical surface centered in $\boldsymbol{x}$.

Appendix A: Incompressibility and Aeroacoustics

In Sec. II we assumed the flow to be incompressible. Here, we examine what happens if the same assumption is made in studying aeroacoustics. For simplicity, we address the issue in the frequency domain, that is, $s = \imath\omega$, for the limited case of a linear problem with no wake, that is, $\sigma_Q = \Delta\varphi = 0$.

In steady-state aerodynamics ($\omega = 0$), we have that the condition $M \ll 1$ is sufficient to allow one to assume that the flow is incompressible because the Prandtl–Glauert transformation becomes irrelevant. In unsteady aerodynamics, one needs both $M \ll 1$ and $\omega\ell/c \ll 1$, where ℓ is the largest distance between two points of the body, for example, wing. In this case, $\omega r/c \le \omega\ell/c \ll 1$ and, therefore, $\exp(-i\omega\breve{\theta}) \approx \exp(-i\omega r/c) \approx 1$, and for $\sigma_Q = \Delta\varphi = 0$, Eq. (38) reduces to Eq. (8).

In aeroacoustics, under the same conditions, for a point that is quite distant from the body, we do not necessarily have $\omega r/c \ll 1$. However, given a point $\boldsymbol{y}_0$ ("average" of the points on the body surface), we have $r - r_0 \le \ell$ ($r_0 = \|\boldsymbol{y}_0 - \boldsymbol{x}\|$), and hence, $\exp[i\omega(\breve{\theta} - r_0/c)] \approx \exp[i\omega(r - r_0)/c] \approx \exp(i\omega\ell/c) \approx 1$, or $\exp(-i\omega\breve{\theta}) \approx \exp(-i\omega r_0/c)$. Hence, the boundary integral representation [Eq. (38)], reduces to

$$\tilde{\varphi}(\boldsymbol{x}) = \exp\left(\frac{-i\omega r_0}{c}\right) \oint_{\mathcal{S}_B} \left[\frac{\partial\tilde{\varphi}}{\partial n_y}\, G - \tilde{\varphi}\, \frac{\partial G}{\partial n_y}\right] \mathrm{d}\mathcal{S}(\boldsymbol{y}) \qquad \text{(A1)}$$

The implication is that, contrary to aerodynamics, in aeroacoustics one may use the incompressible-flow assumption only if a delay r_0/c is included in the solution. On the other hand, the phase difference between the various emitting points may be neglected.

Appendix B: Value of $E(\boldsymbol{x})$ on $\mathcal{S}_B$

The simplest way to obtain the value of $E(\boldsymbol{x})$ for $\boldsymbol{x} \in \mathcal{S}_B$ is to use the Lighthill[53] approach to generalized functions. This consists of introducing the generalized functions, such as the Dirac delta function, as the limit of suitably smooth functions. Specifically, for the three-dimensional case, let us introduce a function $\delta_\epsilon(\boldsymbol{x})$ (near identity), with the following properties: 1) $\delta_\epsilon(\boldsymbol{x})$ is spherically symmetric and is as smooth as needed for the given problem, 2) $\delta_\epsilon(\boldsymbol{x}) = 0$ for $\|\boldsymbol{x}\| > \epsilon$, and 3)

$$\int_{\mathbb{R}^3} \delta_\epsilon(\boldsymbol{x})\, \mathrm{d}\mathcal{V}(\boldsymbol{x}) = 1$$

Such a function exists. Indeed, one such function, which is infinitely differentiable, is given by $\delta_\epsilon(\boldsymbol{x}) = c\exp[1/(r^2 - \epsilon^2)]$ for $r < \epsilon$, and $\delta_\epsilon(\boldsymbol{x}) = 0$ otherwise, with $r = \|\boldsymbol{x}\|$ and

$$c = \frac{1}{4\pi}\int_0^\epsilon \exp\left[\frac{1}{(r^2 - \epsilon^2)}\right] r^2\, \mathrm{d}r$$

so that condition 3 is satisfied. Then, if $\boldsymbol{x}$ is a smooth point of $\mathcal{S}_B$

$$\lim_{\epsilon\to 0}\int_{\mathcal{V}_F} \varphi(\boldsymbol{y})\delta_\epsilon(\boldsymbol{y} - \boldsymbol{x})\, \mathrm{d}\mathcal{V}(\boldsymbol{y}) = \frac{1}{2}\varphi(\boldsymbol{x}) \qquad \text{(B1)}$$

[See comments in brackets following Eq. (8).]

Of course, this implies that one should use an approximate fundamental solution G_ϵ, which satisfies the equation $L^* G_\epsilon = \delta_\epsilon$. For instance, for the Laplacian, using $\delta_\epsilon(\boldsymbol{x}) = 3/4\pi\epsilon^3$ for $\|\boldsymbol{x}\| < \epsilon$, and $\delta_\epsilon(\boldsymbol{x}) = 0$ otherwise, one obtains $G_\epsilon = -1/4\pi r$ ($r = \|\boldsymbol{x} - \boldsymbol{y}\|$) for $r > \epsilon$, and $G_\epsilon = (-3\epsilon^2 + r^2)/8\pi\epsilon^3$ for $r < \epsilon$. In the limit, as ϵ tends to zero, one recovers Eqs. (8) and (9).

Finally, note that $\boldsymbol{x}$ is not always a smooth point of $\mathcal{S}_B$, for instance, for points on a hinge line or on the trailing edge. (See, in particular, the third-order formulation at the end of Appendix D.) Hence, consider the most general case, for which we have

$$\lim_{\epsilon\to 0}\int_{\mathcal{V}_F} \varphi(\boldsymbol{y})\delta_\epsilon(\boldsymbol{y} - \boldsymbol{x})\, \mathrm{d}\mathcal{V}(\boldsymbol{y}) = E(\boldsymbol{x})\varphi(\boldsymbol{x}) \qquad \text{(B2)}$$

where $E(\boldsymbol{x})$ equals the fraction of the support of $\delta_\epsilon(\boldsymbol{x})$ that lies in $\mathcal{V}_F$. This implies $E(\boldsymbol{x}) = 1 - \Omega(\boldsymbol{x})/4\pi$, where $\Omega(\boldsymbol{x})$ is the solid angle connected to $\mathcal{S}_B$ for an observer located in $\boldsymbol{x}$. This includes Eq. (B1) as a particular case.

Appendix C: Fundamental Solutions

In Appendix C, we derive all of the fundamental solutions used in the main body of the paper. Novel results include the critique on the Sommerfeld radiation condition in Appendix C.B, and the fundamental solution for viscous compressible flows in Appendix C.D.

A. Potential Flows

Consider the fundamental solution for the Laplace equation, defined by Eq. (6). Recalling that $\nabla^2 =$ div grad, integrating over a volume $\mathcal{V}$ that includes the point $\boldsymbol{y}$, and using Gauss's theorem, as well as the definition of the Dirac delta function, one obtains

$$\oint_{\mathcal{S}} \frac{\partial G}{\partial n}\,\mathrm{d}\mathcal{S}(\boldsymbol{x}) = 1 \tag{C1}$$

where $\boldsymbol{n}$ is the outer normal to $\mathcal{S} = \partial\mathcal{V}$. Choosing for $\mathcal{S}$ a sphere centered on $\boldsymbol{y}$ and taking into account the spherical symmetry of the problem, one obtains $\partial G/\partial r = 1/4\pi r^2$. Using the boundary condition at infinity for G yields Eq. (7).

B. Acoustics

Next, consider the fundamental solution for acoustics (wave equation), G_A, which is defined by Eq. (14) and by the initial and boundary conditions that follow that equation.

Exploiting the reciprocity theorem [Eq. (72)], one may solve instead the adjoint problem

$$\nabla^2 G_A^* - \frac{1}{c^2}\frac{\partial^2 G_A^*}{\partial t^2} = \delta(\boldsymbol{x}-\boldsymbol{y})\delta(t-\tau) \tag{C2}$$

$[\forall \boldsymbol{x} \in \mathbb{R}^3$ and $\forall t \in (0,\infty)]$, with initial conditions $G_A^*(\boldsymbol{x},0) = \dot{G}_A^*(\boldsymbol{x},0) = 0$ and boundary condition $G_A^*(\boldsymbol{x},t) = o(1)$ at infinity. This is a boundary-value problem in space and an initial-value problem in time. (G_A^* is the causal response to an impulsive point source.) Thus, we take the Fourier transform for the space variables and the Laplace transform for the time variable. Without loss of generality, we may assume that $\boldsymbol{y} = \boldsymbol{0}$ and $\tau = 0^+$. Then,

$$\mathcal{F}\big[\tilde{G}_A^*(\boldsymbol{x},s)\big] = -1/(k^2 + s^2/c^2) \qquad [\mathrm{Re}(s) > 0] \tag{C3}$$

where $\tilde{G}_A^*(\boldsymbol{x},s)$ is the Laplace transform of $G_A^*(\boldsymbol{x},t)$, whereas $\boldsymbol{k} = (k_1,k_2,k_3)$ and $k = \|\boldsymbol{k}\|$.

Next, take the inverse Fourier transform of Eq. (C3). Note that for any function $f(k)$ using spherical coordinates in the $\boldsymbol{k}$ space, with the axis of symmetry directed along $\boldsymbol{x}$, we have

$$\begin{aligned}\mathcal{F}^{-1}f &= \frac{1}{8\pi^3}\int_{-\infty}^{\infty} f(k)\exp(\imath\boldsymbol{x}\cdot\boldsymbol{k})\,\mathrm{d}\mathcal{V}(\boldsymbol{k}) \\ &= \frac{1}{4\pi^2}\int_0^{\infty}\int_{-\pi/2}^{\pi/2}\exp(\imath kr\sin\alpha)f(k)k^2\cos\alpha\,\mathrm{d}\alpha\,\mathrm{d}k \\ &= \frac{1}{2\pi^2 r}\int_0^{\infty} f(k)k\sin(kr)\,\mathrm{d}k \end{aligned} \tag{C4}$$

(denoting by α the angle between $\boldsymbol{k}$ and a plane normal to $\boldsymbol{x}$), where $r = \|\boldsymbol{x}\|$ [Schwartz,[12] p. 204, Eq. (V, 3; 26)]. Hence, noting that $\mathcal{F}\tilde{G}^*$ is an even function of k, one obtains

$$\tilde{G}_A^*(\boldsymbol{x},s) = \frac{\imath}{4\pi^2 r}\int_{-\infty}^{\infty}\frac{1}{k^2 + s^2/c^2}k\exp(\imath kr)\,\mathrm{d}k \tag{C5}$$

Using Jordan's lemma (see Ref. 54) yields

$$\tilde{G}_A^*(\boldsymbol{x},s) = \frac{\imath}{4\pi^2 r}\int_{\mathcal{C}}\frac{1}{k^2 + s^2/c^2}k\exp(\imath kr)\,\mathrm{d}k \tag{C6}$$

where k is now a complex variable, whereas $\mathcal{C}$ is a counterclockwise contour in the complex k plane composed of the segment $(-R, R)$ of the real axis (with R sufficiently large) and of the semicircle in the upper-half k plane, centered in the origin and having radius R.

Recalling that [see Eq. (65)] $\mathrm{Re}(s) > 0$, we see that the pole $k = \imath s/c$ is inside the contour, whereas the pole $k = -\imath s/c$ is outside. Thus,

$$\tilde{G}_A^*(s) = 2\pi\imath\frac{\imath}{4\pi^2 r}\frac{k\exp(\imath kr)}{k + \imath s/c}\bigg|_{k=\imath s/c} = \frac{-1}{4\pi r}\exp\left(\frac{-sr}{c}\right) \tag{C7}$$

Finally, taking the inverse Laplace transform yields the desired result $G_A^*(\boldsymbol{x},t) = -\delta(t - r/c)/4\pi r$. The wave equations is invariant to both time and space shifts; thus, for $\boldsymbol{y} \neq \boldsymbol{0}$ and $\tau \neq 0^+$, one obtains

$$G_A^*(\boldsymbol{x}-\boldsymbol{y}, t-\tau) = (-1/4\pi r)\delta(t - \tau - r/c) \tag{C8}$$

with $r = \|\boldsymbol{x}-\boldsymbol{y}\|$. Equation (C8) implies that the signal due to a disturbance emanating from the emitting point $\boldsymbol{y}$ at time τ arrives at the receiving point $\boldsymbol{x}$ at time $t = r/c > 0$, in line with causality. Using the reciprocity theorem, Eq. (C10), yields Eq. (15), which has anticausal behavior.

Note that the Sommerfeld radiation condition (see Refs. 8 and 9) has not been used. This is because the Laplace transform implies $\mathrm{Re}(s) > 0$. If we had been working in the frequency domain (Fourier transform domain), we would have had $s = \imath\omega$, and both poles would have been on the contour. Then, to get the correct solution, one has to invoke physics, that is, the Sommerfeld condition, because mathematics would require the use of the Cauchy principal value, thereby yielding a physically incorrect solution, the average of the causal and anticausal fundamental solutions.

Finally, we outline the proof of the reciprocity theorem, Eq. (C10). Given an operator L and its adjoint L^*, which by definition satisfies the relationship (Colton and Kress,[8] p. 17)

$$\langle Lu, v\rangle = \langle u, L^*v\rangle \qquad (\forall u, v) \tag{C9}$$

consider G and G^* such that $L^*G(\boldsymbol{x},\boldsymbol{x}_1) = \delta(\boldsymbol{x}-\boldsymbol{x}_1)$ and $LG^*(\boldsymbol{x},\boldsymbol{x}_1) = \delta(\boldsymbol{x}-\boldsymbol{x}_2)$. Then, Eq. (C9) implies $\langle LG^*, G\rangle - \langle G^*, L^*G\rangle = \langle\delta(\boldsymbol{x}-\boldsymbol{x}_2), G(\boldsymbol{x},\boldsymbol{x}_1)\rangle - \langle G^*(\boldsymbol{x},\boldsymbol{x}_2), \delta(\boldsymbol{x}-\boldsymbol{x}_1)\rangle = 0$, or

$$G(\boldsymbol{x}_2,\boldsymbol{x}_1) = G^*(\boldsymbol{x}_1,\boldsymbol{x}_2) \tag{C10}$$

This equation states that G may be obtained from G^* by interchanging $\boldsymbol{x}_1$ with $\boldsymbol{x}_2$ (in our case, $\boldsymbol{x}$ and t with $\boldsymbol{y}$ and τ).

C. Quasi-Potential Subsonic Flows

In this subsection, we denote the quantities in the air frame with the subscript a. Note that the equation for quasi-potential subsonic flows [Eq. (34)] may be obtained from the wave equation (12), using the transformation

$$\boldsymbol{x}_a = \boldsymbol{x} - \boldsymbol{v}_\infty t, \qquad t_a = t \tag{C11}$$

Thus, Eq. (C11) may be used to obtain the fundamental solution of Eq. (34) from the acoustics fundamental solution (15), which in the notations adopted in this subsection is given by

$$G_A = (-1/4\pi r_a)\delta(t_a - \tau_a + r_a/c) \tag{C12}$$

with $r_a = \|\boldsymbol{x}_a - \boldsymbol{y}_a\|$.

To perform the transformation, note that, if $g(t)$ is the argument of the Dirac delta function, then

$$\int_{-\infty}^{\infty} f(t)\delta[g(t)]\,\mathrm{d}t = \sum_k\int_{\mathcal{L}_k}\frac{f}{|\dot{g}|}\delta(g)\,\mathrm{d}g = \sum_k\left[\frac{f}{|\dot{g}|}\right]_{t=t_k} \tag{C13}$$

where t_k are the roots of $g(t) = 0$, whereas $\mathcal{I}_k$ is the image on the g line of an infinitesimal neighborhood of t_i. The preceding equation implies

$$\delta[g(t)] = \sum_k\frac{\delta(t - t_k)}{|\dot{g}|}$$

In our case, from Eqs. (C11) and (C12), we see that $g(t)$ is given by

$$g(t) = t - \tau + \|\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{v}_\infty(t-\tau)\|/c \tag{C14}$$

Note that $g(t) = 0$ implies $\beta^2\theta^2 - 2\boldsymbol{r}\cdot\boldsymbol{v}_\infty\theta/c^2 - r^2/c^2 = 0$, where $\theta = \tau - t = r_a/c \geq 0$, $M = v_\infty/c$, and $\beta^2 = 1 - M^2$, whereas

$\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}$ and $r = \|\boldsymbol{r}\|$. For subsonic flows, this equation has only one positive root, given by

$$\theta = (1/c\beta^2)(\boldsymbol{r} \cdot \boldsymbol{v}_\infty/c + r_\beta) \tag{C15}$$

where $r_\beta^2 = (\boldsymbol{r} \cdot \boldsymbol{v}_\infty/c)^2 + \beta^2 r^2$. In addition,

$$\dot{g} = 1 - \boldsymbol{v}_\infty \cdot (\boldsymbol{r} + \boldsymbol{v}_\infty \theta)/c r_a \tag{C16}$$

Note that $\dot{g}$ coincides with the factor $1 - M_r$ typically used in aeroacoustics. Recalling that $r_a = c\theta$ and combining Eqs. (C15) and (C16), one obtains $r_a \dot{g} = r_a(1 - M_r) = r_\beta$. Finally, combining Eqs. (C12–C16) yields Eq. (36).

D. Viscous Compressible Flows

Next, consider the fundamental solution G_{VC} for viscous compressible flows, which is defined by Eq. (48) and by the initial and boundary conditions that follow that equation. Equation (48) occurs in the boundary integral formulation for compressible viscous flows.[10,11,41] The exact expression of the corresponding fundamental solution is given in Ref. 10, which presents also two approximate but simpler expressions. For simplicity, we discuss one of these simpler expressions, which allows one to examine the characteristics of the solution. Specifically, assuming that ν_2^2 terms are negligible and exploiting the reciprocity theorem [Eq. (C10)], we consider instead the adjoint problem

$$\nabla^2 G^*_{\mathrm{VC}} - \frac{1}{c^2}\left(\frac{\partial}{\partial t} - \nu_2 \nabla^2\right)^2 G^*_{\mathrm{VC}} = \delta(\boldsymbol{x} - \boldsymbol{y})\delta(t - \tau) \tag{C17}$$

where $\boldsymbol{x} \in \mathbb{R}^3$ and $t \in (0, \infty)$, with $G^*_{\mathrm{VC}}(\boldsymbol{x}, t) = 0$ at infinity, and initial conditions $G^*_{\mathrm{VC}}(\boldsymbol{x}, 0) = \dot{G}^*_{\mathrm{VC}}(\boldsymbol{x}, 0) = 0$. Again, this is a boundary-value problem in space and an initial-value problem in time. (G^*_{VC} is the causal response to a point source.) Thus, we take the Fourier transform for the space variables and the Laplace transform for the time variable. Without loss of generality, we may assume that $\boldsymbol{y} = \boldsymbol{0}$ and $\tau = 0^+$. Then,

$$\mathcal{L}\left[\hat{G}^*_{\mathrm{VC}}(\boldsymbol{k}, t)\right] = -1\Big/\left[k^2 + \left(s + \nu_1 k^2\right)^2\Big/c^2\right] \tag{C18}$$

where $\hat{G}^*_{\mathrm{VC}}(\boldsymbol{k}, t)$ is the Fourier transform of $G^*_{\mathrm{VC}}(\boldsymbol{x}, t)$, whereas $k = \|\boldsymbol{k}\|$.

Taking the inverse Laplace transform, one obtains

$$\hat{G}^*_{\mathrm{VC}}(\boldsymbol{k}, t) = -c \exp\left(-\nu_2 k^2 t\right) \sin(kct)/k \tag{C19}$$

(Recall that $\mathcal{L}[\exp(-at)\sin(bt)] = b/[(s + a)^2 + b^2]$.) Taking the inverse Fourier transform and using Eq. (66), we have

$$G^*_{\mathrm{VC}}(\boldsymbol{x}, t) = \frac{-c}{2\pi^2 r}\int_0^\infty \exp\left(-\nu_2 k^2 t\right) \sin(kct)\sin(kr)\,\mathrm{d}k \tag{C20}$$

Next, using Euler formula for $\sin(x)$ and recalling that

$$\int_0^\infty \exp(-u^2)\,\mathrm{d}u = \frac{\sqrt{\pi}}{2}$$

one obtains

$$\int_0^\infty \exp(-c^2u^2)\sin(au)\sin(bu)\,\mathrm{d}u$$

$$= \frac{\sqrt{\pi}}{4c}\left\{\exp\left[-\left(\frac{a-b}{2c}\right)^2\right] - \exp\left[-\left(\frac{a-b}{2c}\right)^2\right]\right\} \tag{C21}$$

and Eq. (C20) reduces to Eq. (49).

Appendix D: Boundary Elements

In this section, we briefly address the issue of the numerical approximation of the boundary integral equations discussed in the main body of this work. The corresponding methodology is known as boundary elements (which is the discretized version of boundary integral equation methods). For the sake of conciseness, only the approach used by this author and his collaborators for the validation of the formulation is addressed.

Consider first potential incompressible flows, for which the boundary integral representation is given by Eq. (8). Let $\mathcal{S}_B$ be divided into triangular and/or quadrilateral elements, and let $\varphi(\boldsymbol{x})$ be approximated by

$$\varphi(\boldsymbol{x}) = \sum_{j=1}^{N_B} \varphi_j M_j(\boldsymbol{x})$$

where $M_j(\boldsymbol{x})$ are suitable interpolation functions discussed at the end of this Appendix, whereas $\varphi_j = \varphi(\boldsymbol{x}_j)$, with $\boldsymbol{x}_j$ the interpolation nodes. A similar expression is used for $\chi(\boldsymbol{x})$, with interpolation functions $L_j(\boldsymbol{x})$, not necessarily equal to $M_j(\boldsymbol{x})$. This yields

$$\varphi(\boldsymbol{x}) = \sum_{j=1}^{N_B} B_j(\boldsymbol{x})\chi_j + \sum_{j=1}^{N_B} C_j(\boldsymbol{x})\varphi_j \tag{D1}$$

where

$$B_j(\boldsymbol{x}) = \int_{\mathcal{S}} L_j(\boldsymbol{y}) G\,\mathrm{d}\mathcal{S}(\boldsymbol{y})$$

$$C_j(\boldsymbol{x}) = -\int_{\mathcal{S}} M_j(\boldsymbol{y})\frac{\partial G}{\partial n_y}\,\mathrm{d}\mathcal{S}(\boldsymbol{y}) \tag{D2}$$

Next, consider the discretization of the boundary integral equation (9). This is based on the so-called collocation method, which consists of satisfying Eq. (84) at N_B prescribed collocation points, which here are assumed to coincide with the interpolation nodes $\boldsymbol{x}_k$, $k = 1, \ldots, N_B$. (See comments at the end of this Appendix.) This yields

$$\frac{1}{2}\varphi_k = \sum_{j=1}^{N_B} B_{kj}\chi_j + \sum_{j=1}^{N_B} C_{kj}\varphi_j \tag{D3}$$

where $B_{kj} = B_j(\boldsymbol{x}_k)$ and $C_{kj} = C_j(\boldsymbol{x}_k)$. The preceding is a system of linear algebraic equations relating the unknowns φ_j to the values of χ_j, known from the boundary condition (3). Once the values of φ_j have been obtained, Eq. (D1) gives $\varphi(\boldsymbol{x})$ in the field, as an infinitely differentiable function of $\boldsymbol{x}$; this implies in particular that one may use derivatives, instead of finite differences, to evaluate $\boldsymbol{v}$ in the field.

Next, consider the discretization for acoustics (wave equation). With the earlier process, Eq. (17) with $\boldsymbol{x} \in \mathcal{S}_B$ and homogeneous initial conditions is approximated by

$$\frac{1}{2}\varphi_k(t) = \sum_{j=1}^{N_B} B_{kj}\chi_j(t - \theta_{kj}) + \sum_{j=1}^{N_B} C_{kj}\varphi_j(t - \theta_{kj})$$

$$+ \sum_{j=1}^{N_B} D_{kj}\dot{\varphi}_j(t - \theta_{kj}) \tag{D4}$$

with B_{kj} and C_{kj} given by Eq. (D2), and a similar definition for D_{kj}. The time discretization is obtained by assuming a linear variation of $\varphi_j(t)$ between t_p and t_{p+1}. This yields a system of recurrent difference equations that may be easily solved step by step. The corresponding equations in the frequency domain are

$$[Y_{kj}(s)]\{\tilde{\varphi}_j\} = [Z_{kj}(s)]\{\tilde{\chi}_j\} \tag{D5}$$

where

$$Y_{kj}(s) = \tfrac{1}{2}\delta_{kj} - (C_{kj} + sD_{kj})\exp(-s\theta_{kj})$$
$$Z_{kj}(s) = B_{kj}\exp(-s\theta_{kj}) \tag{D6}$$

Next, consider the discretization for quasi-potential incompressible flows. With the earlier process, Eq. (31) with $\boldsymbol{x}_k \in \mathcal{S}_B$ is approximated by

$$\frac{1}{2}\varphi_k(t) = \sum_{j=1}^{N_B} B_{kj}\chi_j(t) + \sum_{j=1}^{N_B} C_{kj}\varphi_j(t) + \sum_{n=1}^{N_W} F_{kn}\Delta\varphi_n\big(t - \theta_{C_n}\big) \tag{D7}$$

where θ_{C_n} is the time required for a wake point to be convected from the trailing edge point $\boldsymbol{x}_{\mathrm{TE}_n}$ to the wake point $\boldsymbol{x}_{W_n}$. [Usually $\theta_{C_n} = (x_{W_n} - x_{\mathrm{TE}_n})/v_\infty$.] Also,

$$\Delta\varphi_n(t) = \sum_{j=1}^{N_B} S_{nj}\varphi_j(t) \tag{D8}$$

where S_{nj} is introduced to implement the trailing-edge condition (30). The time discretization is similar to that for acoustics. If a free-wake analysis is being considered, then the vertices of the wake elements are displaced by an amount $\boldsymbol{v}_n\Delta t$, with $\boldsymbol{v}_n$ evaluated by taking the gradient of the discretized boundary integral representation, similar to Eq. (D1). The corresponding equations in the frequency domain are again given by Eq. (D5), where now

$$Y_{kj}(s) = \frac{1}{2}\delta_{kj} - C_{kj} - \sum_{n=1}^{N_W} F_{kn}\exp\big(-s\theta_{C_n}\big)S_{nj}$$
$$Z_{kj} = B_{kj} \tag{D9}$$

Next, consider the discretization for quasi-potential compressible flows. With the earlier process, one obtains

$$\frac{1}{2}\varphi_k(t) = \sum_{j=1}^{N_B} B_{kj}\chi_j(t - \theta_{kj}) + \sum_{j=1}^{N_B} C_{kj}\varphi_j(t - \theta_{kj}) + \sum_{j=1}^{N_B} D_{kj}\dot{\varphi}_j(t - \theta_{kj}) + \sum_{n=1}^{N_W} F_{kn}\Delta\varphi_n(t - \theta_{kn}) + \sum_{n=1}^{N_W} G_{kn}\Delta\dot{\varphi}_n(t - \theta_{kn}) \tag{D10}$$

with $\Delta\varphi_n(t)$ still given by Eq. (D8). The time discretization is similar to that for acoustics. The equations in the frequency domain are again given by Eq. (D5), where now

$$Y_{kj}(s) = \frac{1}{2}\delta_{kj} - (C_{kj} + sD_{kj})\exp(-s\theta_{kj}) - \sum_{n=1}^{N_W}(F_{kn} + sG_{kn})\exp\big[-s\big(\theta_{kn} + \theta_{C_n}\big)\big]S_{nj}$$
$$Z_{kj}(s) = B_{kj}\exp(-s\theta_{kj}) \tag{D11}$$

Finally, the interpolation functions used by the author and his collaborators fall into two categories. In both cases, the surface $\mathcal{S}_B$ is divided into quadrilateral elements $\mathcal{S}_j$, $j = 1, \ldots, N$. In the first formulation (zeroth order[32] the functions $L_j(\boldsymbol{x}) = M_j(\boldsymbol{x})$ are assumed to be piecewise constant, that is, $\varphi(\boldsymbol{x})$ and $\chi(\boldsymbol{x})$ are treated as constant within each element. The elements are approximated with portions of hyperboloidal paraboloids, that is, linear variation along the coordinate lines.[32] The collocation points $\boldsymbol{x}_k$ coincide with the centers of the elements.

In the second formulation (third order[25]), the functions $L_j(\boldsymbol{x})$ and $M_j(\boldsymbol{x})$ are assumed to be piecewise cubic polynomials, that is, $\varphi(\boldsymbol{x})$ and $\chi(\boldsymbol{x})$ have cubic variation within each element; this is obtained by starting from a two-dimensional Hermite interpolation and expressing the derivative through suitable finite difference expressions. The same interpolation is used for the geometry (isoparametric elements). The collocation points coincide with the nodes of the elements.

For quasi-potential flows, this yields a problem at those trailing-edge nodes with $\Delta\varphi \neq 0$, where there exist two unknowns (φ_1 and φ_2), but only one collocation point. This problem is overcome by using, for the same trailing-edge collocation point, two boundary integral equations, that is, 1) Eq. (8) (used for all of the nodes) and 2) the differentiated boundary integral equation, that is, the equation obtained as the limit of the normal derivative of the corresponding boundary integral representation. (This procedure may be thought of as the limiting case of using two collocation points slightly ahead of the trailing edge, an approach previously used by this author.) Incidentally, for collocation points on the trailing edge, $E(\boldsymbol{x}) \neq \frac{1}{2}$ (Appendix B).

Finally, the third-order formulation allows one to explore the necessity of imposing additional the trailing-edge conditions, including the Kutta condition $\Delta p = 0$ at the trailing edge. Numerical evidence[25] indicates that Eq. (30) is the only trailing-edge condition required to solve the problem: The Kutta condition appears to be automatically satisfied.

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